A p-adic cohomological approach to congruences of meromorphic modular forms

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Numerical observation

Let $C/\mathbb{Q}$: $y^2 + xy = x^3 - x^2 - 2x - 1$, $j(C) = -3375$. Consider the meromorphic modular form

$$f(q) \;:=\; \frac{E_4(q)}{j(q) + 3375}$$

where $E_4(q) = 1 + 240\sum_{n\geq 1} \sigma_3(n)\, q^n$ and $j(q)$ is the j-invariant.
For every ordinary prime $p > 3$ and every $\ell \geq 0$:

$$\boxed{a_{np^{\ell+1}}(f) \;\equiv\; \mu^2 \cdot a_{np^{\ell}}(f) \pmod{p^{3\ell}}.}$$

Analytic: $f(q) = \sum_{n\geq 0} a_n(f)\, q^n$, q-expansion.

Arithmetic: μ unit root of $X^2 - a_p(C)X + p$,
$a_p(C) = p + 1 - \#C(\mathbb{F}_p)$.

Goal: Describe the geometric meaning of this congruence.

Let $X/\mathbb{Z}_p$ be a fine smooth moduli scheme of generalized elliptic curves with extra level structure.

Let $\pi\colon \mathcal{E} \to X$ be the universal elliptic curve.

- $\omega := \pi_* \Omega^1_{\mathcal{E}/X}$ Hodge line bundle,
- $\mathcal{V} := R^1\pi_*(\Omega^\bullet_{\mathcal{E}/X})$ relative de Rham.

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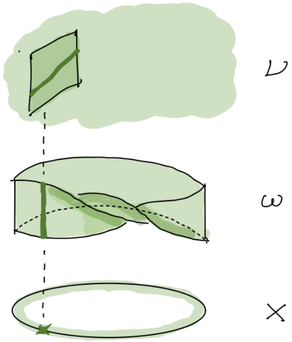
Fiberwise, for $\alpha \in X(\mathbb{Z}_p)$ non-cuspidal rational point:

$\alpha^* \omega = \Gamma(E_\alpha, \Omega^1_{E_\alpha})$ global differentials,

$\alpha^* \mathcal{V} = H^1_{\mathrm{dR}}(E_\alpha)$ de Rham cohomology.

We omitted for simplicity the log-singularities at the cusp



$M_k(X) = \Gamma(X, \omega^{\otimes k})$ *modular forms*.

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The relative de Rham cohomology carries a connection:

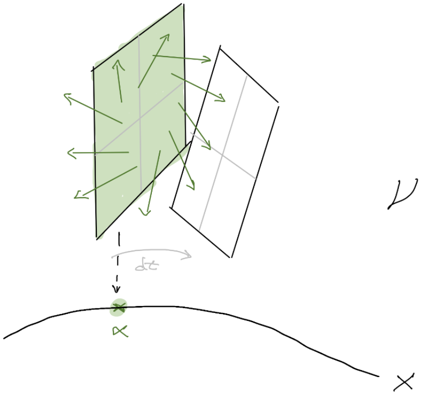
$$\boxed{\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_X(\log C) \quad \textit{Gauss-Manin connection.}}$$

Extend to higher weight:

$$\mathcal{V}_k := \mathrm{Sym}^k \mathcal{V} \supseteq \omega^{\otimes k}.$$

We want to study:

$$\boxed{H^1_{\mathrm{dR}}(X, \mathcal{V}_k) := \mathbb{H}^1\left(X, \left[\mathcal{V}_k \xrightarrow{\nabla} \mathcal{V}_k \otimes \Omega^1_X\right]\right).}$$



§2. Geometric setup (continued)

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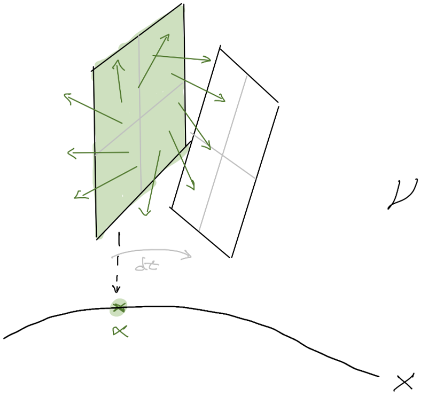
Eichler–Shimura:

$$0 \to M_{k+2}(X) \to H^1_{\mathrm{dR}}(X, \mathcal{V}_k) \to S_{k+2}(X)^\vee \to 0.$$



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Zhang's congruences relate:

- coefficients of meromorphic forms with **prescribed poles**,
- characteristic polynomial of Frobenius on **cohomology of elliptic curves**.

$\rightsquigarrow$ **Residue sequence** for $\alpha \in X(\mathbb{Q}_p)$ corresponding to $E_\alpha/\mathbb{Q}_p$:

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{dR}}(X, \mathcal{V}_k) \arrow[r] & H^1_{\mathrm{dR}}(X \setminus \{\alpha\}, \mathcal{V}_k) \arrow[r, "\mathrm{Res}"] \arrow[d, "\wr"'] & \mathrm{Sym}^k H^1_{\mathrm{dR}}(E_\alpha)(1) \arrow[r] & 0 \\
& & M^{\mathrm{mero},\alpha}_{k+2}(X)/\theta^{k+1} M^{\mathrm{mero},\alpha}_{-k}(X) & &
\end{tikzcd}$$

$\theta := q\frac{d}{dq}$ Serre operator.

- Middle term $\leftrightarrow$ **meromorphic modular forms** with pole at α.
- Right term $\leftrightarrow$ **cohomology of** E_α, with Frobenius eigenvalues α_p, β_p.

The residue sequence is an exact sequence of **filtered φ-modules**.

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Goal. Explicitly characterize the **Frobenius action** on $H^1_{\mathrm{dR}}(X \setminus \{\alpha\}, \mathcal{V}_k)$ in terms of **Hecke operators**

$$\boxed{U_p f(q) = \sum_{n \geq 0} a_{pn}(f)\, q^n, \qquad V_p f(q) = p^k \sum_{n \geq 0} a_n(f)\, q^{pn}.}$$

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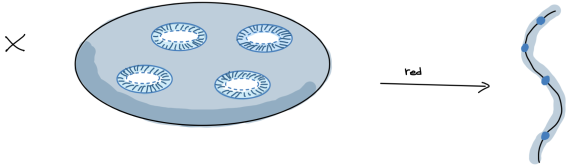
Problem: The U_p-operator lives on modular curves with **p-level** and **moves around the poles!**

Idea : move to **rigid geometry.** Enlarge the set of poles:

$$D = \{\alpha\} \cup \bigcup_{\bar{\beta} \in \mathrm{SS}} \{\beta\}, \qquad V_{D,\lambda} = X^{\mathrm{rig}} \setminus \bigcup_{x \in D} B(x,\, 1-\lambda).$$



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Baldassarri–Chiarellotto '94

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(X \setminus D,\, \mathcal{V}_k) & \xrightarrow{\sim} & H^1_{\mathrm{dR}}(V_{D,\lambda},\, \mathcal{V}_k^{\mathrm{rig}}) \\ \wr\uparrow & & \wr\uparrow \\ M^D_{k+2}(X)/\theta^{k+1} M^D_{-k}(X) & \xrightarrow{\sim} & M^{D,\lambda}_{k+2}/\theta^{k+1} M^{D,\lambda}_{-k} \end{array}$$

$M^{D,\lambda}_{k+2} := \Gamma(V_{D,\lambda}, \omega^{\otimes k,\mathrm{rig}})$ overconvergent modular forms.

§6. The canonical subgroup and Coleman's theorem

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The **canonical subgroup** allows to raise the level:

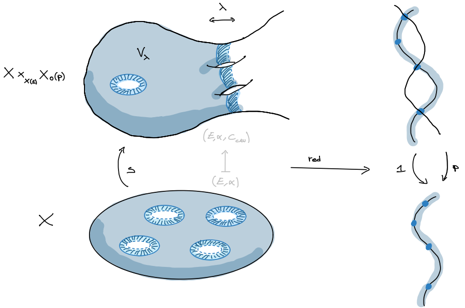
$$E \longmapsto (E,\ C_{E,\mathrm{can}}) \quad \text{on } V_{D,\lambda}.$$

V_p comes from **moduli lift of Frobenius** $E \mapsto E^{(\varphi)}$ on $V_{D,\lambda}$.

Coleman '96

$$[V_p] = \mathrm{Frob}, \quad [U_p] = \mathrm{Ver}, \quad \text{on} \quad H^1_{\mathrm{dR}}(V_{D,\lambda}, \mathcal{V}_k^{\mathrm{rig}}),$$

$$\mathrm{Frob} \circ \mathrm{Ver} \;=\; \mathrm{Ver} \circ \mathrm{Frob} \;=\; p^{k+1}.$$



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$\rightsquigarrow$ The residue sequence is **Frobenius-equivariant**:

$$\boxed{\begin{array}{c} 0 \longrightarrow H^1_{\mathrm{dR}}(X, \mathcal{V}_k) \longrightarrow M^{D,\lambda}_{k+2}/\theta^{k+1} M^{D,\lambda}_{-k} \longrightarrow \bigoplus_{x \in D} \mathrm{Sym}^k H^1_{\mathrm{dR}}(E_x)(1) \longrightarrow 0 \\ \circlearrowleft U_p,\ V_p \qquad\qquad \circlearrowleft \mathrm{Frob}_p. \end{array}}$$

- The Frobenius structure on $H^1_{\mathrm{dR}}(X, \mathcal{V}_k)$ is governed by the **Eichler-Shimura isomorphism**.
- The characteristic polynomial of Frobenius on $\mathrm{Sym}^k H^1_{\mathrm{dR}}(E_x)(1)$ has roots $p\alpha^i\beta^{k-i}$ for α, β roots of $X^2 - a_p(E_x) + p$.

Pulling back a class $[\omega] \in H^1_{\mathrm{dR}}(V_{D,\lambda}, \mathcal{V}_k^{\mathrm{rig}})$ to the cusp $[\infty]$ we obtain an element in $\mathbb{Z}_p[[q]]/\theta^{k+1}(\mathbb{Z}_p[[q]])$.

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Theorem (B. 2026)

Fix $E/\mathbb{Q}$ an elliptic curve. Suppose the characteristic polynomial of Frob_p on $\mathrm{Sym}^k H^1_{\mathrm{dR}}(E)$ is coprime with the characteristic polynomial of Frob_p on $H^1_{\mathrm{dR}}(X, \mathcal{V}_k)$.

Then there exist **meromorphic modular forms** $f_0, \dots, f_k \in M^{j(E)}_{k+2}(SL_2\mathbb{Z})$ with poles at $j(E)$ such that

$$\boxed{a_{mp^\ell}(f_i) \;\equiv\; \alpha^i \beta^{k-i} \cdot a_{mp^{\ell-1}}(f_i) \pmod{p^{(k+1)\ell}}}$$

where α, β are the roots of the Frobenius polynomial on $H^1_{\mathrm{dR}}(E)$.

Zhang's observation $(k = 2)$

The meromorphic modular form f of weight 4 associated with elliptic curve $C/\mathbb{Q}$ coincides with f_0 satisfying congruence $a_{mp^\ell}(f) \;\equiv\; \mu^2 \cdot a_{mp^{\ell-1}}(f_i) \pmod{p^{3\ell}}$.

Further directions (joint with H. Hassan)

- $X = \Gamma\backslash\mathcal{H}_p$ Shimura curve admitting **p-adic uniformization**,
- Study exact sequences of filtered (φ, N)-modules arising from extension by CM points,
- Compute Nekovář **local p-adic height pairing** of such extensions,

$$h : \mathrm{Ext}_{\mathrm{MF}^{\mathrm{ad}}(\varphi,N)}(\mathbb{Q}_p, H^1_{\mathrm{dR}}(X, \mathcal{V}_n)) \times \mathrm{Ext}_{\mathrm{MF}^{\mathrm{ad}}(\varphi,N)}(H^1_{\mathrm{dR}}(X, \mathcal{V}_n), \mathbb{Q}_p(1)) \to \mathbb{Q}_p$$

- Provide p-adic analytic formula for **higher p-adic Green's functions** $\mathcal{G}_k$,

$$\mathcal{G}_k(z_0, w_0) = \int_{z_0}^{\bar{z_0}} \Psi_k(z, w_0, \bar{w_0})\, dz$$

- We expect the value of $\mathcal{G}_k$ at CM point to be the **p-adic logarithm of an algebraic number**.