1. de Rham cohomology and Gauss-Manin connection

Let X a smooth separated scheme of finite type over a Noetherian ring $Spec(R)$, then the sheaf of Kahler differentials is a locally free quasi-coherent module $\Omega^1_{X/R}$. The derivation d defines the complex

$$\Omega^\bullet_{X/R} : 0 \to \mathcal{O}_X \to \Omega^1_{X/R} \to \Omega^2_{X/R} \to \cdots$$

and the cohomology of the complex obtained applying the functor $\Gamma = \Gamma(X, -)$ takes the name of *de Rham cohomology* $H_{dR}(X/R)$. Consider now the relative case $\pi : X \to S$ smooth morphism of varieties over a field k. Then we can consider the sheaf of relative differentials $\Omega^\bullet_{X/S} = \Omega^\bullet_{X/k}/\pi^*\Omega^\bullet_{S/k}$ and define the *relative de Rham cohomology* $\mathcal{H}^\bullet_{dR}(X/S)$ as the right derived hypercomology

$$\mathcal{H}^q_{dR}(X/S) = \mathbb{R}^q\, \pi_*(\Omega^\bullet_{X/S}).$$

From the short exact sequence defining the relative differentials

$$0 \to \pi^*\Omega^1_{S/k} \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0$$

and using the fact that the sheafs are locally free, we obtain a filtration

$$\Omega^\bullet_{X/k} = F^0\Omega^\bullet_{X/k} \supseteq F^1\Omega^\bullet_{X/k} \supseteq F^2\Omega^\bullet_{X/k} \supseteq \cdots$$

with

$$F^i\Omega^\bullet_{X/k}/F^{i+1}\Omega^\bullet_{X/k} = \pi^*\Omega^i_{S/k} \otimes_{\mathcal{O}_X} \Omega^{\bullet - i}_{X/S}$$

The fact that the sheafs are locally free implies that there is a trivializing cover of affine schemes $\{U_i\}_i$ where

$$\Omega^1_{X/k|U_i} \cong \pi^*\Omega^1_{S/k|U_i} \oplus \Omega^1_{X/S|U_i}$$

and then taking exterior powers we obtain

$$\Omega^n_{X/k|U_i} \cong \bigoplus_{i=0}^{n} \left(\pi^*\Omega^i_{S/k|U_i} \otimes \Omega^{n-i}_{X/S|U_i}\right).$$

The filtration is then locally given by

$$F^j\Omega^n_{X/k|U_i} = \bigoplus_{i=0}^{n-j} \left(\pi^*\Omega^i_{S/k|U_i} \otimes \Omega^{n-i}_{X/S|U_i}\right).$$

Since the relative de Rham sheaf is defined to be the sheaf associated to the presheaf that locally assigns

$$U \mapsto \mathbb{H}^q(\pi^{-1}(U), \Omega_{X/S|\pi^{-1}(U)})$$

we can then assume $S = Spec(R)$ to be affine and define the Gauss-Manin connection locally.

Considering the spectral sequence $E_r^{p,q}$ associated to the filtration we have that the first page is given by

$$\begin{array}{cccc}
\Omega^2_{X/S} & \pi^*\Omega^1_{S/k} \otimes_{\mathcal{O}_X} \Omega^2_{X/S} & \pi^*\Omega^2_{S/k} \otimes_{\mathcal{O}_X} \Omega^2_{X/S} & \pi^*\Omega^3_{S/k} \otimes_{\mathcal{O}_X} \Omega^2_{X/S} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\Omega^1_{X/S} & \pi^*\Omega^1_{S/k} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} & \pi^*\Omega^2_{S/k} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} & \pi^*\Omega^3_{S/k} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\mathcal{O}_X & \pi^*\Omega^1_{S/k} & \pi^*\Omega^2_{S/k} & \pi^*\Omega^3_{S/k} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
0 & 0 & 0 & 0
\end{array}$$

The second page is given by

$$0 \longrightarrow H^2_{dR}(X/R) \longrightarrow \Omega^1_{R/k} \otimes_R H^2_{dR}(X/R) \longrightarrow \Omega^2_{R/k} \otimes_R H^2_{dR}(X/R) \longrightarrow \Omega^3_{R/k} \otimes_R H^2_{dR}(X/R)$$

$$0 \longrightarrow H^1_{dR}(X/R) \longrightarrow \Omega^1_{R/k} \otimes_R H^1_{dR}(X/R) \longrightarrow \Omega^2_{R/k} \otimes_R H^1_{dR}(X/R) \longrightarrow \Omega^3_{R/k} \otimes_R H^1_{dR}(X/R)$$

$$0 \longrightarrow H^0_{dR}(X/R) \longrightarrow \Omega^1_{R/k} \otimes_R H^0_{dR}(X/R) \longrightarrow \Omega^2_{R/k} \otimes_R H^0_{dR}(X/R) \longrightarrow \Omega^3_{R/k} \otimes_R H^0_{dR}(X/R)$$

2. Gauss-Manin connection for family of Elliptic Curves

Consider the family of elliptic curves $\mathcal{E} : y^2 = x^3 - t$ over $S = Spec(\mathbb{C}[t, t^{-1}])$. Now recall that a basis for the de Rham cohomology of an elliptic curve is given by

$$H^1_{dR}(\mathcal{E}/S) = \langle \omega, \eta \rangle$$

In particular we have

$$\omega = \frac{dx}{y}, \quad \eta = \frac{xdx}{y}.$$

Given the equation $y^2 = x^3 - t$ we have that $\Omega^1_{\mathcal{E}/S}$ is a free R-module where

$$2ydy - 3x^2dx = 0.$$

Taking $\omega = \frac{2x}{3t}dy - \frac{y}{t}dx$ we can observe that

$$\begin{cases} dx = y\omega, \\ dy = \frac{3x^2}{2}\omega \end{cases}$$

but in the full module $\Omega^1_{\mathcal{E}/\mathbb{C}}$ we have that the relation lifts to

$$2ydy - 3x^2dx + dt = 0.$$

In particular the previous relations become

$$\begin{cases} dx = y\omega - \frac{x}{3t}dt, \\ dy = \frac{3x^2}{2}\omega - \frac{y}{2t}dt. \end{cases}$$

From this description we deduce the following relations between 2-forms

$$\begin{cases} dx \wedge dt = y\omega \wedge dt, \\ dy \wedge dt = \frac{3x^2}{2}\omega \wedge dt, \\ dx \wedge dy = \frac{1}{2}\omega \wedge dt. \end{cases}$$

We can then further differentiate ω in this space obtaining

$$\begin{aligned} d\omega =& \frac{2}{3t}dx \wedge dy - \frac{2x}{3t^2}dt \wedge dy - \frac{1}{t}dy \wedge dx + \frac{y}{t^2}dt \wedge dx = \\ =& \left(\frac{1}{3t} + \frac{x^3}{t^2} + \frac{1}{2t} - \frac{y^2}{t^2}\right)\omega \wedge dt = \\ =& -\frac{1}{6}\omega \wedge dt \end{aligned}$$

The Gauss-Manin connection is the given by

$$\nabla(\omega) = -\frac{1}{6t}\omega \otimes dt.$$

Analogously we can proceed to compute the differential of $x\omega$ in $\Omega_{\mathcal{E}}/\mathbb{C}^2$

$$d(x\omega) = dx \wedge \omega + xd\omega = \frac{2x}{3t}dx \wedge dy - \frac{1}{6t}x\omega \wedge dt = \frac{1}{6t}x\omega \wedge dt.$$

Again the Gauss-Manin connection in this case is given by

$$\nabla(x\omega) = \left(\frac{1}{6t}x\omega\right) \otimes dt.$$

With respect to the basis $\omega, x\omega$ we then have that the connection ∇ is given by

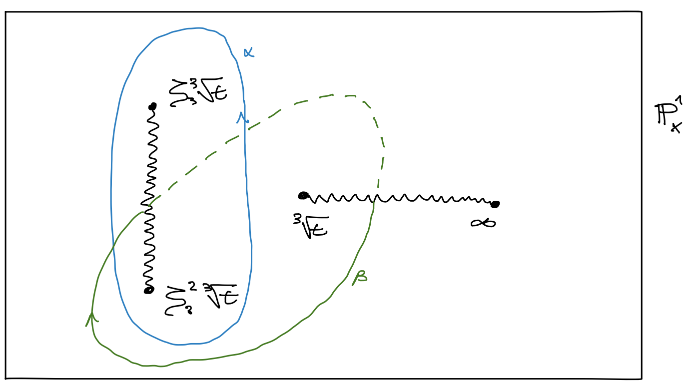
$$\nabla_t = \begin{pmatrix} -1/6t & 0 \\ 0 & 1/6t \end{pmatrix}.$$

Since we are working over $\mathbb{C}$, we can use the de Rham Theorem and shift our point of view on the connection from differential forms to homotopy theory. Recall that we have a perfect pairing given by

$$H^1_{dR}(\mathcal{E}_t, \mathbb{C}) \times H_1(\mathcal{E}_t, \mathbb{R}) \longrightarrow \mathbb{C}$$
$$(\omega, \delta) \longmapsto \int_\delta \omega$$

The fibers of the relative de Rham sheaf $\mathcal{H}_{dR}(\mathcal{E}/S)$ are equal to the 2-dimensional vector space $H^1_{dR}(\mathcal{E}_t, \mathbb{C})$. The Gauss-Manin connection gives then a way to understand how to go from fiber to fiber. For a choice of t, we can picture the elliptic curve in the fiber of t using the double-sheet visualization given by the 2:1 map on the x coordinate

$$\mathcal{E}_t(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$
$$(x, y) \mapsto x$$



Consider a basis for homology given by the paths α, β in the picture where the solid arrows mean that the path is taken only over one sheet while the dotted ones pass to the other sheet. We then have

$$H_1(\mathcal{E}_t, \mathbb{Z}) = \mathbb{Z}\,\alpha + \mathbb{Z}\,\beta.$$

In order to see that α, β form a basis for the homology, we can use the intersection pairing and see that $\langle \alpha, \beta \rangle = 1$.

In order to understand how the 1-chains change from fiber to fiber, we can compute their period, namely

$$\int_\alpha \omega, \quad \int_\alpha x\omega, \quad \int_\beta \omega, \quad \int_\beta x\omega.$$

(1)

$$\int_\alpha \omega = 2\int_{\zeta_3^2 \sqrt[3]{t}}^{\zeta_3 \sqrt[3]{t}} \frac{dx}{\sqrt{x^3 - t}}$$

(2)

$$\int_\alpha \omega = 2\int_{\zeta_3^2 \sqrt[3]{t}}^{\zeta_3 \sqrt[3]{t}} \frac{xdx}{\sqrt{x^3 - t}}$$

(3)
$$\int_\beta \omega = 2\int_{\zeta_3^2\sqrt[3]{t}}^{\sqrt[3]{t}} \frac{dx}{\sqrt{x^3-t}}$$
(4)
$$\int_\beta \omega = 2\int_{\zeta_3^2\sqrt[3]{t}}^{\sqrt[3]{t}} \frac{xdx}{\sqrt{x^3-t}}$$
For the path α consider the change of variables
$$x(z) = \zeta_3^2\sqrt[3]{t} + i\sqrt{3}\sqrt[3]{t}\cdot z$$
we then obtain
$$\omega_\alpha = \int_\alpha \omega = 2\frac{\sqrt{i}t^{-1/6}}{\sqrt[4]{3}}\int_0^1 \frac{dz}{\sqrt{z(z-1)(z+\zeta_3)}}$$
$$\eta_\alpha = \int_\alpha x\omega = 2\frac{\sqrt{i}t^{1/6}}{\sqrt[4]{3}}\int_0^1 \frac{(\zeta_3^2+i\sqrt{3}z)dz}{\sqrt{z(z-1)(z+\zeta_3)}}$$
$$\omega_\beta = \int_\beta \omega = 2\frac{\sqrt{i}t^{-1/6}}{\sqrt[4]{3}}\int_0^{-\zeta_3} \frac{dz}{\sqrt{z(z-1)(z+\zeta_3)}}$$
$$\eta_\beta = \int_\beta x\omega = 2\frac{\sqrt{i}t^{1/6}}{\sqrt[4]{3}}\int_0^{-\zeta_3} \frac{(\zeta_3^2+i\sqrt{3}z)dz}{\sqrt{z(z-1)(z+\zeta_3)}}$$

From which we deduce
$$\frac{d}{dt}\begin{pmatrix}\omega_\alpha & \eta_\alpha\\ \omega_\beta & \eta_\beta\end{pmatrix} = \begin{pmatrix}\omega_\alpha & \eta_\alpha\\ \omega_\beta & \eta_\beta\end{pmatrix}\begin{pmatrix}-1/6 & 0\\ 0 & 1/6\end{pmatrix}.$$

References

[Sil09] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, New York, 2 edition, 2009. First edition published in 1986.