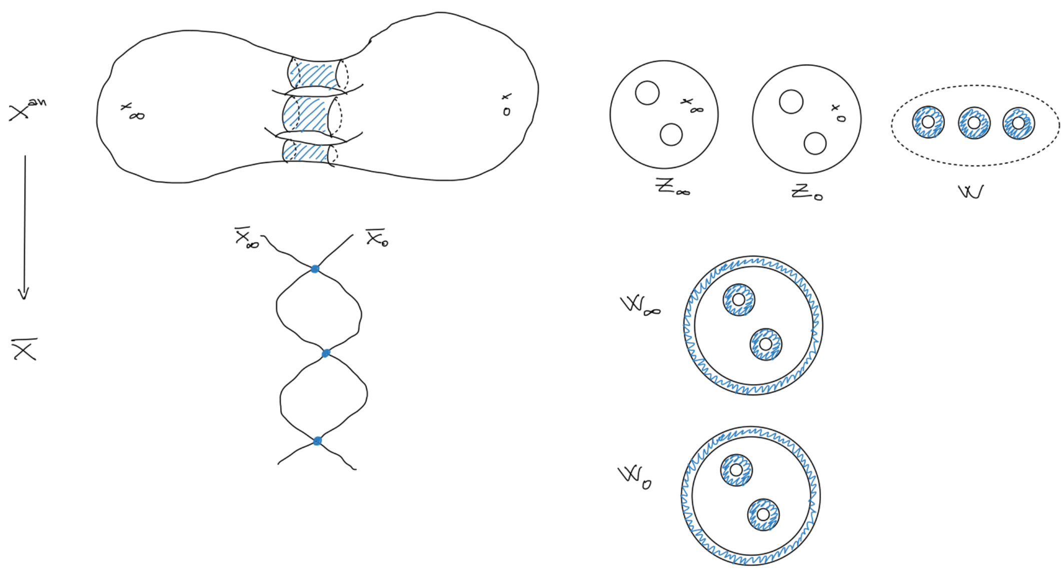
1. Hypercohomology of Modular Curves

Consider $X = X(Np, 2)$ modular curve with Np level structure and full 2 level structure. Let X^{an} be the p-adic rigid analytic curve and consider Z_∞ and Z_0 the ordinary loci containing the cusps $[\infty]$ and $[0]$ that are swap under the Atkin-Lehner involution ω_{Np}. Let W be the union of supersingular annuli. We then have X^{an} is given by the disjoint union

$$X^{an} = Z_\infty \cup W \cup Z_0.$$



It will be useful later to introduce the notation $W_\infty = Z_\infty \cup W$ and $W_0 = Z_0 \cup W$. Let Y the open curve obtained removing the cusps and $\mathcal{E}$ the generalised elliptic curve over X

$$\pi : \mathcal{E} \to X.$$

Let $\mathcal{H}$ be the relative de Rham cohomology with log singularities at the cusps $\mathcal{H}_{dR}(\mathcal{E}/X, \log)$. We then have that $\mathcal{H}$ is a coherent $\mathcal{O}$-mod locally free of rank 2 with fibers $H_{dR}(\mathcal{E}_x/k(x))$. We have a canonical decomposition

$$\mathcal{H} = \underline{\omega} \oplus \underline{\omega}^{-1}$$

with $\underline{\omega} = \pi_* \Omega^1_{\mathcal{E}/X}(\log)$. For k non-negative integer we define the coherent $\mathcal{O}$-mod $\mathcal{H}_k$ to be

$$\mathcal{H}_k := Sym^k(\mathcal{H}) = \underline{\omega}^{-k} \oplus \underline{\omega}^{2-k} \oplus \cdots \underline{\omega}^k.$$

The Gauss-Manin connection $\Delta : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_X$ induces a complex of sheaves

$$\mathcal{H}_k^* : 0 \to \mathcal{H}_k \xrightarrow{\Delta} \mathcal{H}_k \otimes \Omega^1_X \to 0$$

we want to study its hypercohomology $\mathbb{H}^1(X, \mathcal{H}_k^*)$. Consider the covering $\{W_\infty, W_0\}$ and take the double complex $\mathcal{H}^{*,*}$ where the columns are given by Cěch resolution

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{H}_{k|W} & \longrightarrow & (\mathcal{H}_k \otimes \Omega_X^1)_{|W} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{H}_{k|W_\infty} \oplus \mathcal{H}_{k|W_0} & \longrightarrow & (\mathcal{H}_k \otimes \Omega_X^1)_{|W_\infty} \oplus (\mathcal{H}_k \otimes \Omega_X^1)_{|W_0} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{H}_k & \longrightarrow & \mathcal{H}_k \otimes \Omega_X^1 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}$$

We then have the total complex is given by

$$\begin{aligned}
Tot^0(\mathcal{H}^{*,*}) &= \mathcal{H}_k(W_\infty) \oplus \mathcal{H}_k(W_0) \\
Tot^1(\mathcal{H}^{*,*}) &= \mathcal{H}_k(W) \oplus (\mathcal{H}_k \otimes \Omega_X^1))(W_\infty) \oplus (\mathcal{H}_k \otimes \Omega_X^1))(W_0) \\
Tot^2(\mathcal{H}^{*,*}) &= (\mathcal{H}_k \otimes \Omega_X^1))(W)
\end{aligned}$$

The 0th hypercohomology group can then be directly read as the group of global horizontal sections

$$\mathbb{H}^0(X, \mathcal{H}_k^*) = \{\eta \in \mathcal{H}_k(X) : \nabla \eta = 0\}.$$

To compute the first group, we have that the cocycles and coboundaries are given by

$$\begin{aligned}
Z^1 &= \{(\eta, \xi_\infty, \xi_0) \in Tot^1(\mathcal{H}^{*,*}) : \nabla \eta = \xi_{\infty|W_\infty} - \xi_{0|W_\infty}\} \\
B^1 &= \{(\eta_\infty - \eta_0, \nabla \eta_\infty, \nabla \eta_0) \in Tot^0(\mathcal{H}^{*,*}) : \eta_\infty \in \mathcal{H}_k(W_\infty), \eta_0 \in \mathcal{H}_k(W_0)\}
\end{aligned}$$