

# A $p$ -adic cohomological approach to congruences of meromorphic modular forms

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## The motivating congruences

**Zhang's numerical congruences** [Zha25]. For an elliptic curve  $C/\mathbb{Q}$  with good reduction at  $p$  and  $a_p(C) = p + 1 - |C(\mathbb{F}_p)|$ , Zhang observed:

$$a_p\left(\frac{E_4}{j-j(C)}\right) \equiv a_p(C)^2 \pmod{p}$$

and systematic congruences linking **Fourier coefficients** of meromorphic modular forms to **Frobenius eigenvalues of the pole** elliptic curves. For  $\tau \in \mathbb{C}$ ,  $\text{Im}(\tau) > 0$  and  $q = e^{2\pi i \tau}$ , we have

$$E_4(q) := 1 + 240 \sum_{n \geq 1} \left( \sum_{d|n} d^3 \right) q^n \text{ the weight four Eisenstein series,}$$

$j(\tau)$  the  $j$ -function associated to the elliptic curve  $C/(\mathbb{Z} + \tau\mathbb{Z})$ .

**Goal.** Give a uniform  $p$ -adic cohomological interpretation of these congruences.

**Key tool:** Rigid residue sequence and log-crystalline cohomology to produce a Frobenius-stable  $\mathbb{Z}_p$ -lattice in the rigid cohomology.

## Geometric setup

Consider  $X$  a fine moduli scheme of generalized elliptic curves with extra level structure admitting a **smooth model** over  $\mathbb{Z}_p$ . Let  $\pi : \mathcal{E} \rightarrow X$  be the universal elliptic curve and  $C$  be the cuspidal subgroup. Consider the following coherent sheaves

- $\omega := \pi_* \Omega_{\mathcal{E}/X}^1(\log \pi^{-1}(C))$  the Hodge line bundle from the universal object
- $(\mathcal{H}, \nabla) := R^1 \pi_* (\Omega_{\mathcal{E}/X}^1)$  the rank-2 vector bundle from *relative de Rham cohomology*, with *Gauss-Manin connection* fitting into exact sequence

$$0 \rightarrow \omega \rightarrow \mathcal{H} \rightarrow \omega^{-1} \rightarrow 0,$$

- $\mathcal{H}_k = \text{Sym}^k \mathcal{H}$  with the induced Hodge filtration:

$$\mathcal{H}_k = \text{Fil}^0 \mathcal{H}_k \supset \cdots \supset \text{Fil}^{k+1} \mathcal{H}_k = 0, \quad \text{Gr}^i \mathcal{H}_k \cong \omega^{k-2i}$$

**Modular forms:**

$$M_k(X) = \Gamma(X, \omega^k), \quad S_k(X) = \Gamma(X, \omega^{k-2} \otimes \Omega_X^1)$$

**Eichler-Shimura exact sequence:**

$$0 \rightarrow M_{k+2}(X) \rightarrow H_{\text{dR}}^1(X, \mathcal{H}_k) \rightarrow S_{k+2}(X)^\vee \rightarrow 0$$

**Serre derivative**

$$0 \rightarrow \omega_{-k} \xrightarrow{\theta^{k+1}} \omega_{k+2} \rightarrow \mathcal{H}_k \otimes_X \Omega_X^1(\log C) / \nabla_k(\mathcal{H}_k) \rightarrow 0$$

where  $\theta = q \frac{d}{dq}$  on  $q$ -expansion.

## The residue sequence with hidden structure

Fix an effective divisor  $S = \sum_i \alpha_i$  on  $X \setminus C$ , with  $\alpha_i \in X(\mathbb{Z}_p)$  having distinct reductions mod  $p$ . We have an exact sequence

$$0 \longrightarrow H_{\text{dR}}^1(X^{\text{rig}}, \mathcal{H}_k^{\text{rig}}) \longrightarrow H_{\text{dR}}^1(X \setminus S, \mathcal{H}_k^{\text{rig}}) \xrightarrow{\text{Res}} \bigoplus_{\alpha \in S} \text{Sym}^k H_{\text{dR}}^1(\mathcal{E}_\alpha)[1] \longrightarrow 0$$

$$\downarrow \wr$$

$$M_{k+2}^{\text{mero}, S} / \theta^{k+1} M_{-k}^{\text{mero}, S}$$

The residue sequence is exact in the category of **filtered- $\varphi$ -modules** [CI10].

The comparison result of Baldassarri-Chiarellotto [BC94] allows to remove small disks at the poles in the rigid category

$$H_{\text{dR}}^1(X \setminus S, \mathcal{H}_k) \cong H_{\text{dR}}^1(V_\lambda, \mathcal{H}_k^{\text{rig}}) \cong M_{k+2}^{\dagger, \lambda} / \theta^{k+1} M_{-k}^{\dagger, \lambda}.$$

## An overconvergent idea

To explicitly extract information about the Frobenius structure we apply Katz-Lubin theory of **canonical subgroup** [Kat73].

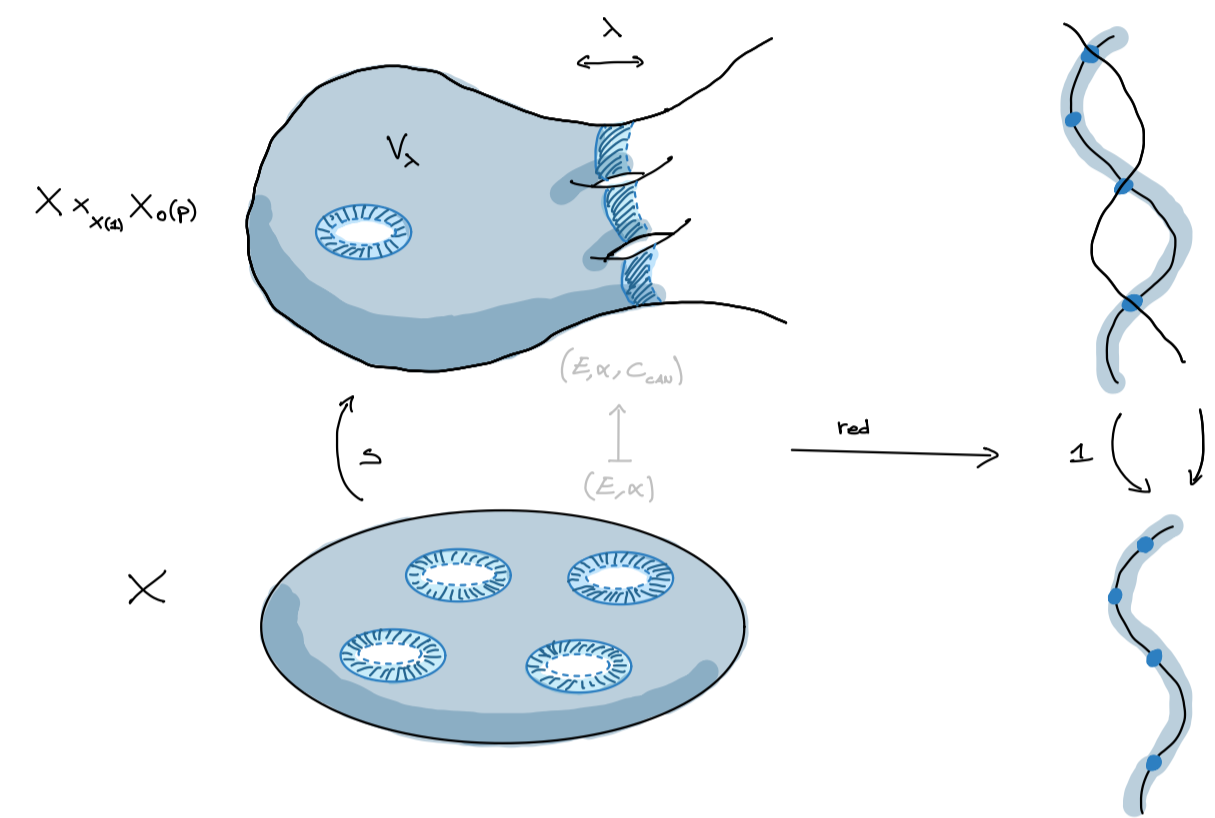


Figure 1. Canonical lift

These spaces are naturally endowed with two modular operators  $V_p, U_p$  acting on  $q$ -expansions  $f(q) = \sum_{n \geq 0} a_n(f) q^n$  as

$$V_p f(q) = \sum_{n \geq 0} a_n(f) q^{np}, \quad U_p f(q) = \sum_{n \geq 0} a_{np}(f) q^n$$

and a normalized eigenform for  $U_p$  has eigenvalue  $a_p(f)$ . These operators induce endomorphisms  $\text{Frob}, \text{Ver}$  on  $H_{\text{dR}}^1(V_\lambda, \mathcal{H}_k^{\text{rig}})$  [Col96] such that

$$\text{Frob} \circ \text{Ver} = \text{Ver} \circ \text{Frob} = p^{k+1} \text{ on } H_{\text{dR}}^1(V_\lambda, \mathcal{H}_k^{\text{rig}}).$$

## Theorem 1 (B. 2026)

Let  $P(X)$  (resp.  $Q(X)$ ) be the characteristic polynomial of Frobenius on  $\text{Sym}^k H_{\text{dR}}^1(\mathcal{E}_\alpha)[1]$  (resp.  $H_{\text{dR}}^1(X^{\text{rig}}, \mathcal{H}_k^{\text{rig}})$ ), and set  $P(X)Q(X) = \sum_{i=0}^M e_i X^i$  with  $M = \dim H_{\text{dR}}^1(X^{\text{rig}}, \mathcal{H}_k^{\text{rig}}) + k + 1$ .

For every  $f \in M_{k+2}^{\text{mero}, \{\alpha\}}$ :

$$\sum_{i=0}^M e_{M-i} p^{M-i} U_p^i(f) \in \theta^{k+1} \left( M_{-k, \lambda}^{\dagger, \{\alpha\}} \right).$$

*Proof idea:* Annihilate  $H_{\text{dR}}^1(V_\lambda, \mathcal{H}_k^{\text{rig}})$  via  $P \cdot Q(\text{Frob})$ , enlarge  $S$  to supersingular locus to make the Frobenius explicit, then substitute  $\text{Ver}^M \circ \text{Frob}^i \leftrightarrow p^{M-i} U_p^i$ .

## Frobenius lattice and $q$ -expansions

Consider the infinitesimal neighborhood  $\text{Spf}(\mathbb{Z}_p[[q^{1/N}]])$  of the cusp  $[\infty]$  of the formal model  $\mathfrak{X}$ . The fiber product with the universal elliptic curve gives rise to the **Tate curve**

$$\begin{array}{ccc} \text{Tate}(q^{1/N}) & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \text{Spf}(\mathbb{Z}_p[[q^{1/N}]]) & \xrightarrow{\iota} & \mathfrak{X}. \end{array}$$

Thanks to the formal smooth lift of the variety we can pull back the *log-crystalline cohomology* at infinity to obtain integral  $q$ -expansion description

$$H_{\log-\text{crys}}^1(\overline{(\mathcal{X}, \overline{S})}/\mathbb{Z}_p, \mathcal{H}_k) \rightarrow \mathbb{Z}_p[[q^{1/N}]] / \theta^{k+1} (\mathbb{Z}_p[[q^{1/N}]]).$$

## Theorem 2 (B. 2026)

Let  $e_i$  be the coefficients of the polynomial  $P(X) \cdot Q(X)$  as before. Then for every

$$f \in \text{Im}(H_{\log-\text{crys}}^1(\overline{(\mathcal{X}, \overline{S})}/\mathbb{Z}_p, \mathcal{H}_k) \rightarrow H_{\log-\text{crys}}^1(\overline{(\mathcal{X}, \overline{S})}/\mathbb{Z}_p, \mathcal{H}_k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H_{\text{dR}}^1(V_\lambda, \mathcal{H}_k^{\text{rig}}))$$

we have:

$$\sum_{i=0}^M e_{M-i} p^{M-i} a_{np^i}(f) \equiv 0 \pmod{p^{l(k+1)}}.$$

## An explicit example

Let  $C/\mathbb{Q}$  be the CM elliptic curve  $y^2 + xy = x^3 - x^2 - 2x - 1$ , with CM by  $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$  and  $j(C) = -3375$ .

Zhang's basis for  $\text{Sym}^2 H_{\text{dR}}^1(C) \hookrightarrow H_{\text{dR}}^1(V_\lambda, \mathcal{H}_2^{\text{rig}})$ :

$$\begin{aligned} f_1 &= \frac{E_4}{j+3375}, \\ f_2 &= 19 \cdot \frac{E_4}{j+3375} - 91125 \cdot \frac{E_4}{(j+3375)^2}, \\ f_3 &= 1399 \cdot \frac{E_4}{j+3375} - 19008675 \cdot \frac{E_4}{(j+3375)^2} + 54251268750 \cdot \frac{E_4}{(j+3375)^3} \end{aligned}$$

For every ordinary prime  $p > 3$  and  $\ell \geq 0$ , letting  $u_p(C)$  be the  $p$ -adic unit root of  $X^2 - a_p(C)X + p$ :

$$a_{np^{\ell+1}}(f_i) \equiv \mu_i \cdot a_{np^\ell}(f_i) \pmod{p^{3\ell}}$$

where  $\mu_1 = u_p(C)^2$ ,  $\mu_2 = p$ ,  $\mu_3 = p^2 u_p(C)^{-2}$ .

**Cohomological interpretation.** The three forms  $f_1, f_2, f_3$  represent the Frobenius eigenspace  $\text{Sym}^2 H_{\text{dR}}^1(C)[1]$  inside  $H_{\text{dR}}^1(V_\lambda, \mathcal{H}_2^{\text{rig}})$ , and the congruences follow directly from Theorem 2 applied to  $P(X) = \det(\text{Frob} - X \mid \text{Sym}^2 H_{\text{dR}}^1(C)[1])$ .

## Further directions: towards semistable reduction and Gross-Zagier conjecture

In a joint work with Hazem Hassan, we are extending the framework to the **semistable case** for a Shimura curve admitting  $p$ -adic uniformization in the sense of Čerednik-Drinfeld. The residue sequence at CM divisors produces natural extensions of filtered  $(\varphi, N)$ -modules. We provide a  $p$ -adic analytic formula for the  $p$ -adic height pairing of these extensions that can be regarded as a **higher  $p$ -adic Green's function**. We expect the value to be the  $p$ -adic logarithm of an algebraic number, a  $p$ -adic analogue of the Gross-Zagier formula.

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