Crystalline Cohomology

Notes for the study group on Weil cohomologies, Leiden, Fall 2025

Paolo Bordignon

8 and 15 December 2025

Abstract

Notes for two talks on crystalline cohomology. The first part motivates the theory through Grothendieck's infinitesimal site [Gro68] and introduces divided powers and the crystalline site [Ber74, BO78]. The second part reviews the comparison with de Rham cohomology and introduces Witt vectors and Illusie's de Rham–Witt complex [Ill79]. Good general references are the survey [CL98] and the book [BO78]; for a modern account of de Rham–Witt complexes see [BLM21].

Contents

1 Crystalline cohomology I: the crystalline site **1**
1.1 Introduction and motivation 1
1.2 Infinitesimal site 4
1.3 Crystalline site 7

2 Crystalline cohomology II: de Rham–Witt complexes **9**
2.1 Review of the crystalline site 9
2.2 Witt vectors, Witt rings, Witt schemes 10
2.3 The de Rham–Witt complex 12

1 Crystalline cohomology I: the crystalline site

1.1 Introduction and motivation

Recap on Weil cohomologies

Let X be smooth and proper over $\mathbb{Z}_p$, and fix an embedding $\iota\colon \mathbb{Q}_p \hookrightarrow \mathbb{C}$. We have

$$\begin{array}{ccccc} X_{\mathbb{C}} & \longrightarrow & X & \longleftarrow & X_{\mathbb{F}_p} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{Z}_p & \longleftarrow & \operatorname{Spec}\mathbb{F}_p \end{array}$$

and to $X_{\mathbb{C}}(\mathbb{C})$ we associate the complex manifold X^{an}. So far we have seen:

- $H_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z})$;
- $H_{\mathrm{dR}}(X_\square)$ for $\square \in \{\emptyset, \mathbb{C}, \mathbb{F}_p\}$, and $H_{\mathrm{dR}}(X^{\mathrm{an}})$;
- $H_{\text{ét}}(X_\square, \mathbb{Z}_\ell) = \varprojlim_n H_{\text{ét}}(X_\square, \mathbb{Z}/\ell^n\mathbb{Z})$, for $\ell \neq p$.

ℓ-adic cohomology is endowed with Frob_p on $X_{\mathbb{F}_p}$, giving the zeta function via

$$P_i(t) = \det\big(1 - \mathrm{Frob}_p\ t \;\big|\; H^i_c(X_{\mathbb{F}_p} \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)\big).$$

Remark 1.1. Proper base change holds for arbitrary torsion coefficients, but smooth base change and $\mathbb{A}^1$-homotopy invariance require the coefficients to be prime to the characteristic. For $\ell = p$ the theory fails (in a spectacular way). For instance, $H^1_{\text{ét}}(\mathbb{A}^1_{\overline{\mathbb{F}}_p}, \mathbb{Z}/p\mathbb{Z}) \cong \overline{\mathbb{F}}_p[t]/(1 - \mathrm{Frob}_p)\overline{\mathbb{F}}_p[t]$ is infinite.

To see what goes wrong for proper varieties, let $\overline{X} = X_{\overline{\mathbb{F}}_p}$ and $d = \dim \overline{X}$. Consider the Artin–Schreier exact sequence on $\overline{X}_{\text{ét}}$

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{O}_{\overline{X}} \xrightarrow{1-\mathrm{Frob}_p} \mathcal{O}_{\overline{X}} \longrightarrow 0,$$

with $\mathrm{Frob}_p \colon a \mapsto a^p$ the absolute Frobenius. Recall that

$$H^i_{\text{ét}}(\overline{X}, \mathcal{O}_{\overline{X}}) = H^i_{\text{Zar}}(\overline{X}, \mathcal{O}_{\overline{X}}) \qquad \text{for every } i \geq 0$$

(this holds for any quasi-coherent sheaf on any scheme), and these groups vanish for $i > d$ by Grothendieck's vanishing theorem. Since $H^d(\overline{X}, \mathcal{O}_{\overline{X}})$ is a finite-dimensional $\overline{\mathbb{F}}_p$-vector space and Frob_p acts p-linearly on it, $1 - \mathrm{Frob}_p$ is surjective there. The long exact sequence therefore gives

$$H^i(\overline{X}, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \forall\, i > d \qquad \rightsquigarrow \qquad H^i(\overline{X}, \mathbb{Z}_p) = 0 \quad \forall\, i > d.$$

In particular, if $d \geq 1$ then $H^{2d}(\overline{X}, \mathbb{Q}_p) = 0$, so there is no Poincaré duality and $H^\bullet_{\text{ét}}(\overline{X}, \mathbb{Q}_p)$ is not a Weil cohomology. (Over $\mathbb{F}_p$ itself the bound becomes $i > d + 1$, since $\mathbb{F}_p$ has p-cohomological dimension 1: e.g. $H^1(\operatorname{Spec} \mathbb{F}_p, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.)

De Rham cohomologies

Grothendieck's idea (1966–68, see [Gro68]): define the p-adic cohomology of X in terms of the de Rham cohomology of lifts to characteristic 0.

Consider k a perfect field of characteristic $p > 0$ and $W(k) = W$ its ring of Witt vectors. Let X be smooth and proper over k, and assume

$$H^0(X, T_X) = 0 \qquad \text{and} \qquad H^2(X, T_X) = 0.$$

The first condition makes the deformation functor of X over W pro-representable, the second makes it unobstructed. Hence there is a *universal formal deformation*

$$f \colon \mathcal{X} \longrightarrow \operatorname{Spf} W[[t_1, \ldots, t_r]] = S, \qquad r = \dim_k H^1(X, T_X),$$

i.e. for every scheme Z/W smooth and proper lifting X, $Z \times_W \operatorname{Spec} k \cong X$, there is a point $x \in S(W)$ (a continuous W-algebra map $x \colon W[[t]] \to W$, so $x(t_j) \in pW$) such that

$$\widehat{Z} \cong \mathcal{X} \times_S W = \mathcal{X} \otimes_{W[[t]]} (W, x),$$

where $\widehat{Z}$ is the p-adic completion of Z. Let $\mathcal{H}^i = R^i f_* \Omega^\bullet_{\mathcal{X}/S}$ be the relative de Rham cohomology. Suppose $\mathcal{H}^i$ is free of finite rank h and compatible with base change:

$$H^i_{\text{dR}}(Z/W) = x^* \mathcal{H}^i = \mathcal{H}^i \otimes_{W[[t]]} (W, x).$$

$\mathcal{H}^i$ is endowed with a natural integrable connection, the *Gauss–Manin connection*

$$\nabla_{\text{GM}} \colon \mathcal{H}^i \longrightarrow \mathcal{H}^i \otimes \widehat{\Omega}^1_{S/W},$$

where $\widehat{\Omega}^1_{S/W}$ is the module of continuous differentials, free on $dt_1, \ldots, dt_r$.

Let Z_1, Z_2 be two lifts of X/k corresponding to $x, y\colon W[[t]] \to W$. Since all $x(t_j), y(t_j) \in pW$, we automatically have $x \equiv y \bmod p$. Assume $p \geq 3$ (for $p = 2$, assume $x \equiv y \bmod 4$). Then we have an isomorphism

$$\chi(x,y)\colon \mathcal{H}^i \otimes (W,x) \longrightarrow \mathcal{H}^i \otimes (W,y), \qquad \chi(x,y)(s) = \sum_{n\in\mathbb{N}^r} \frac{(x(t)-y(t))^n}{n!}\Big(\nabla_{\mathrm{GM}}\big(\tfrac{\partial}{\partial t}\big)^n s\Big)(y),$$

for s a section of $\mathcal{H}^i$, with multi-index notation $(x(t)-y(t))^n = \prod_j (x(t_j)-y(t_j))^{n_j}$, $n! = \prod_j n_j!$ and $\nabla(\partial/\partial t)^n = \prod_j \nabla(\partial/\partial t_j)^{n_j}$. The series converges because $v_p\big(p^m/m!\big) \geq m - \frac{m-1}{p-1} \to \infty$ for $p \geq 3$. These isomorphisms satisfy $\chi(x,y)\chi(y,z) = \chi(x,z)$ and $\chi(x,x) = \mathrm{id}$. In particular, $H^i_{\mathrm{dR}}(Z_1/W) \cong H^i_{\mathrm{dR}}(Z_2/W)$ canonically, even though Z_1 and Z_2 need not be isomorphic.

Exercise 1.2. Prove that $\chi(x,y)$ is a well-defined isomorphism, first assuming ∇ nilpotent (so that the sum is finite).

Hint: fix a basis $e_1, \dots, e_h$ of $\mathcal{H}^i$ and write $\nabla(e_i) = \sum_j \theta_{ij}\, dt_j$ with $\theta_{ij} \in \mathcal{H}^i$, so that

$$\nabla(fe_i) = df \cdot e_i + f\sum_j \theta_{ij}\, dt_j, \qquad \nabla = d + \sum_j M_j\, dt_j, \quad M_j \in \mathrm{End}(\mathcal{H}^i) = M_h(W[[t]]).$$

Integrability means $\partial_j M_k - \partial_k M_j + [M_j, M_k] = 0$ for all j, k. Parallel transport: find a section m such that

- $\nabla(m) = 0$,
- $x^* m = s$.

Note that m involves the terms $(t - x(t))^n/n!$, so in general it does not exist in $\mathcal{H}^i$ itself, but only over the divided power envelope of the ideal $(t_j - x(t_j))_j$. It can nevertheless be evaluated at y, because $y(t_j) - x(t_j) \in pW$ and pW has divided powers. This is a first appearance of divided powers.

Remark 1.3 (Algebraic vs. analytic de Rham). For X smooth and proper over $\mathbb{C}$ we get X^{an}, a complex analytic variety, and

$$\underline{\mathbb{C}}_X \longrightarrow \Omega^\bullet_{X^{\mathrm{an}}}$$

is a quasi-isomorphism by the *holomorphic Poincaré lemma* (the C^∞ version, according to de Rham, was proved by Volterra in 1889 but attributed to Poincaré). This induces an isomorphism

$$H^\bullet(X^{\mathrm{an}}, \underline{\mathbb{C}}_X) \cong H^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}) = \mathbb{H}^\bullet(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}).$$

On the algebraic side, the Poincaré lemma fails Zariski-locally.

Example 1.4. Let $U = D(1+t) = \operatorname{Spec}\mathbb{C}[t, (1+t)^{-1}]$, a Zariski open neighbourhood of 0 in $\mathbb{A}^1_{\mathbb{C}}$, and $\omega = \dfrac{dt}{1+t} \in \Omega^1(U)$. The form ω is closed but has no primitive on any Zariski open neighbourhood of zero, since $\log(1+t)$ is not algebraic. However, $\log(1+t)$ does live in $\mathbb{C}[[t]]$: a primitive exists *formally*, i.e. on the infinitesimal neighbourhoods of 0.

Grothendieck's idea: introduce infinitesimal deformations [Gro68].

1.2 Infinitesimal site

> My propositions are elucidatory in this way: he who understands me finally recognizes them as senseless, when he has climbed out through them, on them, over them. (He must so to speak throw away the ladder, after he has climbed up on it.) He must surmount these propositions; then he sees the world rightly.
>
> L. Wittgenstein, *Tractatus Logico-Philosophicus*, 6.54 (1921; trans. C. K. Ogden, 1922)

We follow [Gro68, §4]; see also [CL98]. Let $f\colon X \to S$ be a morphism of schemes (no assumption on regularity or characteristic).

Definition 1.5. The *infinitesimal site* $\mathrm{Inf}(X/S)$ is defined as follows.

- **Objects:** $(U \hookrightarrow T)$ with
  - $U \subset X$ a Zariski open subset;
  - $U \hookrightarrow T$ a nilpotent closed immersion of S-schemes, i.e. $\mathcal{I} = \ker(\mathcal{O}_T \to \mathcal{O}_U)$ satisfies $\mathcal{I}^n = 0$ for some $n > 0$.
- **Morphisms:** $(U,T) \to (U',T')$ are commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{u} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{t} & T' \end{array}$$

  with $u\colon U \hookrightarrow U'$ the inclusion of open subsets of X and $t\colon T \to T'$ an S-morphism.
- **Coverings:** $\{(U_i,T_i) \to (U,T)\}_i$ is a covering if each $t_i\colon T_i \to T$ is an open immersion, $T = \bigcup T_i$ is a covering, and $U_i = U \times_T T_i$.

Proposition 1.6. *A sheaf of sets on* $\mathrm{Inf}(X/S)$ *can be identified with a system of Zariski sheaves of sets* $\{\mathcal{F}_{(U,T)}\}_{(U,T)}$ *on* T*, for each* (U,T) *in* $\mathrm{Inf}(X/S)$*, together with, for each* $(u,t)\colon (U,T) \to (U',T')$*, a homomorphism*

$$\rho_t\colon t^{-1}\mathcal{F}_{(U',T')} \longrightarrow \mathcal{F}_{(U,T)}$$

satisfying

- *the transitivity condition for* $(U,T) \to (U',T') \to (U'',T'')$*;*
- ρ_t *is an isomorphism for* $t\colon T \to T'$ *an open immersion.*

Proof. Exercise. □

Example 1.7. The *structure sheaf* $\mathcal{O}_{X_{\mathrm{inf}}}$ is given by $(U,T) \longmapsto \Gamma(T,\mathcal{O}_T)$.

> Un cristal possède deux propriétés caractéristiques : la rigidité, et la faculté de croître [...]
>
> A. Grothendieck, letter to J. Tate, Pisa, May 1966

Definition 1.8. A *crystal* of $\mathcal{O}_{X_{\mathrm{inf}}}$-modules is a sheaf $\mathcal{F}$ of $\mathcal{O}_{X_{\mathrm{inf}}}$-modules such that for any morphism $(U,T) \xrightarrow{(u,t)} (U',T')$ the homomorphism

$$\rho_t \colon t^*\mathcal{F}_{(U',T')} \longrightarrow \mathcal{F}_{(U,T)}$$

is an isomorphism.

Definition 1.9. Let $\Delta^n_{X/S}$ be the n-th infinitesimal neighbourhood of the diagonal in $X \times_S X$, with projections $p_1^{(n)}, p_2^{(n)} \colon \Delta^n_{X/S} \to X$. A *stratification* on an $\mathcal{O}_X$-module $\mathcal{F}$ is a compatible system of isomorphisms $\varepsilon_n \colon p_2^{(n)*}\mathcal{F} \xrightarrow{\sim} p_1^{(n)*}\mathcal{F}$, $n \geq 0$, with $\varepsilon_0 = \mathrm{id}$, satisfying the cocycle condition on the infinitesimal neighbourhoods of the diagonal in $X \times_S X \times_S X$.

The datum ε_1 alone is equivalent to a connection on $\mathcal{F}$; the cocycle condition at order two forces it to be integrable.

Proposition 1.10. *Let X be smooth over S.*

(a) The category of crystals of $\mathcal{O}_{X_{\mathrm{inf}}}$-modules is equivalent to the category of $\mathcal{O}_X$-modules with a stratification relative to S.

(b) If moreover S is a $\mathbb{Q}$-scheme, every integrable connection extends uniquely to a stratification. Hence the category of crystals of $\mathcal{O}_{X_{\mathrm{inf}}}$-modules is equivalent to the category of $\mathcal{O}_X$-modules with integrable connection relative to S.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module with a stratification (ε_n), and let (U,T) be an object of $\mathrm{Inf}(X/S)$:

$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & \overset{g}{\nwarrow} & \downarrow \\ S & \longleftarrow & T \end{array}$$

Since X is smooth (hence formally smooth) over S, a retraction $g \colon T \to X$ with $g|_U$ the inclusion exists locally on T, e.g. whenever T is affine. Define $\mathcal{F}_{(U,T)} = g^*\mathcal{F}$. If $h_1, h_2 \colon T \to X$ are two such retractions and $\mathcal{I}^{n+1} = 0$, then $(h_1,h_2) \colon T \to X \times_S X$ factors through the n-th infinitesimal neighbourhood of the diagonal,

$$T \xrightarrow{\pi} \Delta^n_{X/S} \longrightarrow X \times_S X,$$

and $\pi^*\varepsilon_n$ gives an isomorphism $h_2^*(\mathcal{F}) \cong h_1^*(\mathcal{F})$. The cocycle condition ensures that these isomorphisms are compatible, so the local definitions glue. The rest of the proof of (a) is an exercise.

For (b), given an integrable connection ∇, in local coordinates $t_1, \dots, t_r$ on X with $\tau_j = 1 \otimes t_j - t_j \otimes 1$ one sets

$$\varepsilon(1 \otimes s) = \sum_{n \in \mathbb{N}^r} \frac{\tau^n}{n!} \otimes \nabla\big(\tfrac{\partial}{\partial t}\big)^n s,$$

which makes sense since $n!$ is invertible; integrability gives the cocycle condition. □

Remark 1.11. In characteristic p, part (b) fails: a stratification is the same as a module over Grothendieck's full ring of differential operators, which contains operators such as $\frac{1}{p!}\frac{\partial^p}{\partial t^p}$ that are not generated by derivations. An integrable connection only provides the action of derivations. On the crystalline site, crystals will instead correspond to *quasi-nilpotent* integrable connections [BO78].

Remark 1.12. $\mathrm{Inf}(X/S)$ has two pathologies:

- it has no final object (unlike Top, Zar, ét, etc.);
- there is no canonical way to pull back thickenings.

We therefore move to the *infinitesimal topos* $(X/S)_{\mathrm{inf}} \coloneqq \mathrm{Sh}(\mathrm{Inf}(X/S))$.

- The final object in $(X/S)_{\mathrm{inf}}$ is given by $\{*\}$, the constant sheaf with value a singleton. It is not representable.
- Pullback of thickenings: given

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

we get a morphism of topoi $f_{\mathrm{inf}}\colon (X'/S')_{\mathrm{inf}} \to (X/S)_{\mathrm{inf}}$, i.e. a pair of adjoint functors $(f_{\mathrm{inf}}^{-1}, f_{\mathrm{inf}*})$. Let (U,T) in $\mathrm{Inf}(X/S)$ be seen as a sheaf, and define

$$\widetilde{f^{-1}(U,T)}\colon (U',T') \longmapsto \begin{cases} \emptyset & \text{if } f(U') \not\subset U, \\ \{h\colon T' \to T \text{ making } (*) \text{ commute}\} & \text{otherwise,} \end{cases}$$

where $(*)$ is the diagram

$$\begin{array}{ccccc} U' & & \xrightarrow{f} & & U \\ & \searrow & & \swarrow & \\ \downarrow & & T' & \dashrightarrow & T \\ & \swarrow & & \searrow & \\ S' & & \longrightarrow & & S \end{array} \qquad (*)$$

The sheaf $\widetilde{f^{-1}(U,T)}$ is not representable in general.

Let $\widetilde{X}$ be the sheaf on $\mathrm{Inf}(X/S)$

$$\widetilde{X}\colon (U,T) \longmapsto \{g\colon T \to X \ S\text{-morphism with } g|_U = (U \subset X)\},$$

and for $m \geq 0$ let $\widetilde{X}^m \coloneqq \widetilde{X} \times \cdots \times \widetilde{X}$ be its $(m+1)$-fold product.

Remark 1.13. The sheaf $\widetilde{X}^m$ can be described as a colimit of representable sheaves:

$$\widetilde{X}^m = \varinjlim_i \widetilde{\Delta^i_{X/S}(m+1)},$$

where $\Delta^i_{X/S}(j)$ is the i-th infinitesimal neighbourhood of the diagonal $X \to X \times_S \cdots \times_S X$ (j factors).

Fact 1.14. *If X is smooth over S, then $\widetilde{X} \to \{*\}$ is a covering of the final object in $(X/S)_{\mathrm{inf}}$. Indeed, retractions $T \to X$ exist locally on T by formal smoothness.*

We can therefore form the *Čech–Alexander complex* $\check{\mathrm{C}}\mathrm{A}^\bullet\big((\widetilde{X} \to \{*\}), \mathcal{O}_{X_{\mathrm{inf}}}\big)$, with

$$\check{\mathrm{C}}\mathrm{A}^m\big((\widetilde{X} \to \{*\}), \mathcal{O}_{X_{\mathrm{inf}}}\big) = \varprojlim_i P^i_{X/S}(m+1),$$

where $P^i_{X/S}(j)$ is the structure sheaf of $\Delta^i_{X/S}(j)$, viewed as a sheaf of rings on X (it has the same underlying space).

Proposition 1.15. *Let X be smooth over S. Then*

$$H^\bullet_{\mathrm{inf}}(X, \mathcal{O}_{X_{\mathrm{inf}}}) \cong \mathbb{H}^\bullet\big(X_{\mathrm{Zar}}, \check{\mathrm{C}}\mathrm{A}^\bullet((\widetilde{X} \to \{*\}), \mathcal{O}_{X_{\mathrm{inf}}})\big).$$

Theorem 1.16 ([Gro68])**.** *If S is a $\mathbb{Q}$-scheme and X is smooth over S, there is a canonical isomorphism*

$$H^\bullet_{\mathrm{inf}}(X, \mathcal{O}_{X_{\mathrm{inf}}}) \cong H^\bullet_{\mathrm{dR}}(X/S).$$

Remark 1.17. The proof of the theorem is based on the *infinitesimal Poincaré lemma*: the formal de Rham complex of $\mathcal{O}_X[[\tau_1, \dots, \tau_r]]$ is a resolution of $\mathcal{O}_X$ when S is a $\mathbb{Q}$-scheme. It shows that the sheaves in the Čech–Alexander complex can be resolved via a de Rham sequence (using linearising functors).

Note 1.18. The constructions before Proposition 1.10(b) and the last theorem work in arbitrary characteristic, but the infinitesimal Poincaré lemma does not. If S is an $\mathbb{F}_p$-scheme, then in $k[[\tau]]$ the element τ^p is closed but not constant, and $\tau^{p-1}d\tau$ is closed but not exact. $\rightsquigarrow$ We need to find a way to extend the Poincaré lemma to the p-adic setting. $\rightsquigarrow$ *Berthelot's idea (1974) [Ber74]:* introduce *divided power structures* (suggested by Grothendieck [Gro68], inspired by Roby [Rob65], Lazard [?] '65).

1.3 Crystalline site

For divided powers and the crystalline site we follow [BO78, §§3–5]; see also [Ber74, CL98].

Definition 1.19. Let A be a ring and $I \subset A$ an ideal. A *divided power structure* (PD structure) on I is a collection of maps $\gamma_n \colon I \to A$ for $n \geq 0$ such that, for all $x, y \in I$ and $\lambda \in A$,

i) $\gamma_0(x) = 1$, $\gamma_1(x) = x$;

ii) $\gamma_n(x) \in I$ if $n \geq 1$;

iii) $\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$;

iv) $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$;

v) $\gamma_n(x)\gamma_m(x) = \binom{m+n}{n}\gamma_{n+m}(x)$;

vi) $\gamma_m(\gamma_n(x)) = \dfrac{(mn)!}{m!\,(n!)^m}\gamma_{mn}(x)$ for $n \geq 1$.

Remark 1.20. From the definition it follows that $n!\,\gamma_n(x) = x^n$. $\rightsquigarrow$ These are exactly the elements that are "needed" to integrate: $\int x^{n-1}dx = x^n/n = (n-1)!\,\gamma_n(x)$.

Example 1.21.

- If A is a $\mathbb{Q}$-algebra, every ideal has a unique PD structure, $\gamma_n(x) = x^n/n!$.
- Let V be a DVR of mixed characteristic $(0, p)$, with uniformizer π and absolute ramification index e. Then the maximal ideal πV has a PD structure iff $e \leq p-1$. The ideal pV always has one, given by $\gamma_n(px) = \frac{p^n}{n!}x^n$; in particular pW has a unique PD structure.
- Let A be a ring. The *free PD-algebra* in r variables $A\langle x_1, \dots, x_r\rangle$ has the following structure:
  - it is free as an A-module on the symbols $x_1^{[i_1]} \cdots x_r^{[i_r]}$;
  - PD structure $\gamma_n(x_i^{[1]}) = x_i^{[n]}$ (morally $x_i^{[n]} = x_i^n/n!$);
  - multiplication $x_i^{[m]}x_i^{[n]} = \binom{m+n}{n}x_i^{[m+n]}$;
  - graded ring structure $\bigoplus \Gamma^n$, with Γ^n generated by $x_1^{[i_1]} \cdots x_r^{[i_r]}$ with $i_1 + \cdots + i_r = n$, and PD ideal $I = \bigoplus_{n \geq 1} \Gamma^n$.

For $A\langle t_1,\dots,t_n\rangle$ let $\omega^\bullet_{A\langle t_1,\dots,t_n\rangle} = A\langle t_1,\dots,t_n\rangle \otimes_{A[t_1,\dots,t_n]} \Omega^\bullet_{A[t_1,\dots,t_n]/A}$ be the *PD de Rham complex*, with the differential determined by $d\big(t_j^{[k]}\big) = t_j^{[k-1]}\,dt_j$.

Lemma 1.22 (Crystalline Poincaré lemma)**.** *Let A be any ring. The PD de Rham complex $\omega^\bullet_{A\langle t_1,\dots,t_n\rangle}$ is a resolution of A.*

Proof. For $n=1$, the complex $A\langle t\rangle \to A\langle t\rangle\,dt$ is given by

$$\sum_{k\geq 0} a_k t^{[k]} \longmapsto \sum_{k\geq 1} a_k t^{[k-1]}\,dt, \qquad \text{i.e. } d(\gamma_k(t)) = \gamma_{k-1}(t)\,dt.$$

This map is surjective with kernel A, so we have the exact sequence

$$0 \longrightarrow A \longrightarrow A\langle t\rangle \longrightarrow A\langle t\rangle\,dt \longrightarrow 0.$$

Suppose by induction that $A \to \omega^\bullet_{A\langle t_1,\dots,t_{n-1}\rangle}$ is a quasi-isomorphism. We have $\omega^\bullet_{A\langle t_1,\dots,t_n\rangle} \cong \omega^\bullet_{A\langle t_1,\dots,t_{n-1}\rangle} \otimes_A \omega^\bullet_{A\langle t_n\rangle}$, and the terms of $\omega^\bullet_{A\langle t_n\rangle}$ are free, hence flat, A-modules. Thus

$$A \otimes_A \omega^\bullet_{A\langle t_n\rangle} \longrightarrow \omega^\bullet_{A\langle t_1,\dots,t_{n-1}\rangle} \otimes_A \omega^\bullet_{A\langle t_n\rangle}$$

is a quasi-isomorphism. Combined with the case $n=1$, this shows that $A \to \omega^\bullet_{A\langle t_1,\dots,t_n\rangle}$ is a quasi-isomorphism. □

Remark 1.23. Compare with the Note above: in $\mathbb{F}_p\langle t\rangle$ we now have $t^{p-1}dt = d\big((p-1)!\,t^{[p]}\big)$.

We can then try to modify the infinitesimal site to incorporate PD structures $\rightsquigarrow$ Berthelot's 1974 thesis [Ber74].

Definition 1.24. Let k be a perfect field of characteristic $p>0$, $W=W(k)$ its ring of Witt vectors, $W_n = W/p^n$ with the canonical PD structure γ on pW_n, and X a k-scheme. Define the *crystalline site* $\mathrm{Cris}(X/W_n)$:

- **Objects:** commutative diagrams

$$\begin{array}{ccc} U & \xhookrightarrow{\;i\;} & V \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} W_n \end{array}$$

  with $U \subset X$ Zariski open and $i\colon U \hookrightarrow V$ a *PD-thickening*, i.e. a closed immersion of W_n-schemes with $\ker(\mathcal{O}_V \to \mathcal{O}_U)$ endowed with a PD-structure δ compatible with γ.
- **Morphisms:** $(U,V,\delta) \to (U',V',\delta')$ are commutative diagrams with $U \subset U'$ the inclusion of open subsets of X and $V \to V'$ a morphism compatible with the PD-structures.
- **Coverings:** as for the infinitesimal site.

Remark 1.25. We can define and characterise sheaves as for the infinitesimal site (Proposition 1.6).

Definition 1.26. *Crystalline cohomology* is defined by

$$H^i(X/W_n) = H^i\big((X/W_n)_{\mathrm{crys}}, \mathcal{O}_{X/W_n}\big), \qquad H^i(X/W) = H^i\big(R\varprojlim_n R\Gamma((X/W_n)_{\mathrm{crys}}, \mathcal{O}_{X/W_n})\big).$$

If X is smooth and proper over k, the groups $H^i(X/W_n)$ are W_n-modules of finite length. The Mittag-Leffler condition then holds, and $H^i(X/W) = \varprojlim_n H^i(X/W_n)$.

Theorem 1.27 ([Ber74, BO78])**.** *Let X be smooth over k. If Z/W_n is a smooth lift of X, i.e. Z is smooth over W_n with $Z \times_{W_n} \operatorname{Spec} k \cong X$, then crystalline cohomology is canonically isomorphic to de Rham cohomology:*

$$H^i(X/W_n) \cong H^i_{\mathrm{dR}}(Z/W_n).$$

2 Crystalline cohomology II: de Rham–Witt complexes

2.1 Review of the crystalline site

We follow [BO78]; see also [CL98]. Let k be a perfect field, $W = W(k)$ its ring of Witt vectors with the canonical PD structure γ on pW, and X a W-scheme on which p is locally nilpotent (e.g. X/k). Let $S = \operatorname{Spec} W_n$ or $S = \operatorname{Spf} W$. The *crystalline site* $\mathrm{Crys}(X/S)$ has

- **objects:** triples (U, T, δ), with $U \subset X$ an open subset, T an S-scheme on which p is locally nilpotent, and $U \hookrightarrow T$ a closed immersion whose ideal carries a PD structure δ compatible with γ;
- **morphisms:** commutative diagrams, with $U \subset U'$ the inclusion and $T \to T'$ a PD-morphism;
- **coverings:** induced by Zariski coverings of T.

The *structure sheaf* $\mathcal{O}_{X/S}$ of $\mathrm{Crys}(X/S)$ is defined by

$$\mathcal{O}_{X/S}\big((U,T,\delta)\big) = \Gamma(T, \mathcal{O}_T).$$

For every W_n-algebra B and ideal $J \subset B$ with divided powers δ, define the *module of differentials compatible with divided powers* as the module of W_n-linear differentials quotiented by the relations

$$d(\delta_n(x)) = \delta_{n-1}(x)\, dx \qquad \forall\, x \in J,\ n \geq 1.$$

Write $\Omega_{B,\delta}$ for this module.

We can describe it via the diagonal embedding. Given an object (U, T, δ) of $\mathrm{Crys}(X/S)$, let $(U, T(1), \delta(1))$ be the product of (U, T, δ) with itself: the scheme $T(1)$ is the divided power envelope, compatible with δ and γ, of $U \hookrightarrow T \times_S T$, and $\delta(1)$ is the induced PD structure. The diagonal $\Delta\colon T \to T(1)$ is a closed immersion with ideal $\mathcal{I}$. We then define the *sheaf of differentials* on the crystalline site by

$$\Omega^1_{X/S}\big((U,T,\delta)\big) = \Gamma\big(T(1), \mathcal{I}/\mathcal{I}^{[2]}\big),$$

with $\mathcal{I}^{[2]}$ the second divided power of $\mathcal{I}$ (note that $\mathcal{I}/\mathcal{I}^{[2]}$ is supported on $\Delta(T)$). Let $\Omega^i_{X/S}$ be the i-th exterior power of $\Omega^1_{X/S}$.

Proposition 2.1. *There is a natural morphism of topoi*

$$u_{X/S}\colon (X/S)_{\mathrm{crys}} \longrightarrow X_{\mathrm{Zar}}$$

given by:

i) for $\mathcal{F} \in (X/S)_{\mathrm{crys}}$ and $j\colon U \hookrightarrow X$ an open immersion,

$$(u_{X/S*}\mathcal{F})(U) = \Gamma\big((U/S)_{\mathrm{crys}}, j^*_{\mathrm{crys}}\mathcal{F}\big);$$

ii) for $E \in X_{\mathrm{Zar}}$ and $(U, T, \delta) \in \mathrm{Crys}(X/S)$,

$$(u^{-1}_{X/S}E)(U,T,\delta) = \Gamma(U, E|_U).$$

Since $\Gamma((X/S)_{\mathrm{crys}}, -) = \Gamma(X_{\mathrm{Zar}}, -) \circ u_{X/S*}$, we have canonical isomorphisms

$$R\Gamma(X_{\mathrm{Zar}}, Ru_{X/S*}\mathcal{F}) \cong R\Gamma\big((X/S)_{\mathrm{crys}}, \mathcal{F}\big), \qquad H^i(X_{\mathrm{Zar}}, Ru_{X/S*}\mathcal{F}) \cong H^i\big((X/S)_{\mathrm{crys}}, \mathcal{F}\big).$$

Assume X embeds into Z smooth over S, with $j\colon X \hookrightarrow Z$ a closed immersion (such embeddings always exist locally on X). In general the ideal $\ker(\mathcal{O}_Z \to \mathcal{O}_X)$ does not have a PD structure. Consider $\overline{Z}$, the PD-envelope of X in Z compatible with γ, so that $X \hookrightarrow \overline{Z} \to Z$. For a crystal $\mathcal{E}$ of $\mathcal{O}_{X/S}$-modules there is a unique integrable, quasi-nilpotent connection

$$d\colon \mathcal{E}_{\overline{Z}} \longrightarrow \mathcal{E}_{\overline{Z}} \otimes_{\mathcal{O}_Z} \Omega^1_{Z/S}.$$

Theorem 2.2 ([BO78])**.** *For $\mathcal{E}$ a quasi-coherent crystal of $\mathcal{O}_{X/S}$-modules there is a canonical isomorphism*

$$Ru_{X/S*}\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\overline{Z}} \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/S}, \qquad H^\bullet_{\mathrm{crys}}(X/S, \mathcal{E}) \xrightarrow{\sim} \mathbb{H}^\bullet\big(X_{\mathrm{Zar}}, \mathcal{E}_{\overline{Z}} \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/S}\big).$$

Corollary 2.3. *If Z/W_n is a smooth lift of X, then the ideal $p\mathcal{O}_Z$ of X in Z carries a PD structure compatible with γ (because Z is flat over W_n). Hence $\overline{Z} = Z$ and*

$$H^\bullet_{\mathrm{crys}}(X/W_n) \cong H^\bullet_{\mathrm{dR}}(Z/W_n).$$

Remark 2.4 (Frobenius structure). Let $F\colon X \to X$ be the absolute Frobenius of X/k and $\sigma\colon W_n \to W_n$ the Frobenius of W_n, which lifts the Frobenius of k. The square formed by (F, σ) commutes, so by functoriality of crystalline cohomology we get a *σ-semilinear* endomorphism

$$F^*\colon H^\bullet(X/W_n) \longrightarrow H^\bullet(X/W_n).$$

If X lifts to Z/W_n, then we get an action of Frobenius on $H_{\mathrm{dR}}(Z/W_n)$, even though in general the Frobenius does not lift to Z.

2.2 Witt vectors, Witt rings, Witt schemes

The general idea of Witt vectors is to construct a lift of a characteristic p object to characteristic 0.

Let $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[T_1, \dots, T_n]$ be affine space. $\mathbb{A}^n$ represents the functor $R \mapsto R^n$ on rings, and it is naturally endowed with a ring structure: for $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{A}^n(R)$,

$$(x)_i + (y)_i \coloneqq (x + y)_i, \qquad (x)_i \cdot (y)_i = (x \cdot y)_i \qquad \text{(componentwise)}.$$

Denote by $\mathbb{E}^n$ the scheme in rings $\mathbb{A}^n$ (scheme + ring functor structure). Then there exists a unique ring scheme structure on $\mathbb{A}^n$ making the morphism of schemes (the *ghost map*)

$$(\varphi_i)\colon \mathbb{A}^n \longrightarrow \mathbb{E}^n, \qquad \varphi_i\colon (x_1, \dots, x_n) \longmapsto \sum_{j=1}^{i} p^{j-1} x_j^{p^{i-j}} = x_1^{p^{i-1}} + p\, x_2^{p^{i-2}} + \cdots + p^{i-1} x_i$$

a homomorphism of schemes of rings. Uniqueness holds because the ghost map is injective on R-points for R torsion-free.

Definition 2.5. $\mathbb{W}_n$ is the scheme $\mathbb{A}^n$ endowed with this ring structure, called the *Witt scheme*.

In particular, there exist $S_n, P_n \in \mathbb{Z}[X_1, \dots, X_n, Y_1, \dots, Y_n]$ such that

$$a + b = (a_1 + b_1,\ S_2(a, b), \dots), \qquad a \cdot b = (a_1 b_1,\ P_2(a, b), \dots).$$

For example,

- $S_2 = X_2 + Y_2 + \dfrac{X_1^p + Y_1^p - (X_1 + Y_1)^p}{p}$,

- $P_2 = X_1^p Y_2 + X_2 Y_1^p + p X_2 Y_2$.

Remark 2.6. The definition of the ring structure comes with a universal property. Let Y be a ring scheme (resp. group scheme) that is flat over $\mathbb{Z}$. Then $f\colon Y \to \mathbb{W}_n$ is a morphism of ring schemes (resp. group schemes) iff $(\varphi_i) \circ f\colon Y \to \mathbb{E}^n$ is. This uses that the ghost map is injective on torsion-free rings, and it does *not* apply over $\mathbb{F}_p$, where $\varphi_i \equiv x_1^{p^{i-1}}$.

- *Restriction homomorphism* of ring schemes:

$$R_n\colon \mathbb{W}_{n+1} \to \mathbb{W}_n, \qquad (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

- *Verschiebung homomorphism* of (additive) group schemes, characterised by $\varphi_i(Vx) = p\,\varphi_{i-1}(x)$:

$$V_n\colon \mathbb{W}_n \to \mathbb{W}_n, \qquad (x_1, \dots, x_n) \mapsto (0, x_1, \dots, x_{n-1}).$$

- *Frobenius*: over $\mathbb{Z}$ there is a ring homomorphism $F\colon \mathbb{W}_{n+1} \to \mathbb{W}_n$ characterised by $\varphi_i(Fx) = \varphi_{i+1}(x)$. Modulo p it is given by $x_i \mapsto x_i^p$, so in characteristic p we get the *Frobenius homomorphism* of ring schemes

$$F_n\colon \mathbb{W}_{n,\mathbb{F}_p} \to \mathbb{W}_{n,\mathbb{F}_p}, \qquad (x_1, \dots, x_n) \mapsto (x_1^p, x_2^p, \dots, x_n^p).$$

  Equivalently, for an $\mathbb{F}_p$-algebra A, $F_n = \mathbb{W}_n(\mathrm{Frob}_A)$.

Exercise 2.7. Verify with the universal property that R_n and V_n are homomorphisms, and that F (over $\mathbb{Z}$) is a ring homomorphism. Show that F reduces modulo p to $(x_i) \mapsto (x_i^p)$.

Definition 2.8. The limit ring scheme of *Witt vectors* is given by

$$\mathbb{W} = \varprojlim\big(\cdots \to \mathbb{W}_{n+1} \xrightarrow{R_n} \mathbb{W}_n \to \cdots\big).$$

Proposition 2.9. *In characteristic p, i.e. on $\mathbb{W}(A)$ for every $\mathbb{F}_p$-algebra A, one has*

$$F \circ V = V \circ F = [p].$$

Proof. Since F acts coordinatewise in characteristic p, $FV(x) = (0, x_1^p, x_2^p, \dots) = VF(x)$, so it remains to show $VF = [p]$. We work universally: let $x = (x_1, x_2, \dots)$ with x_i indeterminates, and set, in $\mathbb{W}(\mathbb{Z}[x_1, x_2, \dots])$,

$$h = (h_1, h_2, \dots) = [p]\,x - (0, x_1^p, x_2^p, \dots).$$

It suffices to show that p divides h_i for all i: reducing modulo p then gives $[p]x = VF(x)$ universally. Since the ghost map is a ring homomorphism, the terms telescope and

$$\varphi_i(h) = p\,\varphi_i(x) - \varphi_i(0, x_1^p, x_2^p, \dots) = \sum_{j=1}^{i} p^j x_j^{p^{i-j}} - \sum_{j=1}^{i-1} p^j x_j^{p^{i-j}} = p^i x_i.$$

We show $p \mid h_i$ by induction.

- $i = 1$: $h_1 = \varphi_1(h) = p x_1$.
- $i = 2$: $\varphi_2(h) = h_1^p + p h_2 = p^2 x_2$, so $h_2 = p x_2 - p^{p-1} x_1^p$.
- General i: if $h_j = p g_j$ for $j < i$, then $p^{i-1} h_i = p^i x_i - \sum_{j<i} p^{j-1+p^{i-j}} g_j^{p^{i-j}}$. Since $p^m \geq m+1$ for $m \geq 1$, every exponent satisfies $j - 1 + p^{i-j} \geq i$, so $p^i \mid p^{i-1} h_i$, i.e. $p \mid h_i$. □

Remark 2.10. From this it follows that $p = V(1) = (0, 1, 0, \dots)$ in $\mathbb{W}(A)$ for every $\mathbb{F}_p$-algebra A (this fails in $\mathbb{W}(\mathbb{Z})$).

Definition 2.11. For any ring A there is a natural multiplicative (but not additive) map

$$[-]\colon A \longrightarrow \mathbb{W}(A), \qquad a \longmapsto (a, 0, 0, 0, \dots),$$

called the *Teichmüller map*.

Definition 2.12. Let X be a scheme over $\operatorname{Spec} k$, k a perfect field of characteristic p. Define a sheaf of rings $W\mathcal{O}_X$ whose sections on an open $U \subset X$ are given by sequences $(a_1, a_2, a_3, \dots)$ with $a_i \in \Gamma(U, \mathcal{O}_X)$:

$$\Gamma(U, W\mathcal{O}_X) = \mathbb{W}\big(\Gamma(U, \mathcal{O}_X)\big).$$

We can as well define $W_n\mathcal{O}_X = W\mathcal{O}_X / V^n W\mathcal{O}_X$, so that $W\mathcal{O}_X = \varprojlim_n W_n\mathcal{O}_X$.

Remark 2.13. $(X, W\mathcal{O}_X)$ can be thought of as a formal lift to characteristic 0 of the scheme $(X, \mathcal{O}_X)$, but only in a weak sense. We have $W\mathcal{O}_X / VW\mathcal{O}_X = \mathcal{O}_X$, whereas $W\mathcal{O}_X / pW\mathcal{O}_X \neq \mathcal{O}_X$ unless X is perfect, because $p = VF$, so $pW\mathcal{O}_X = V(FW\mathcal{O}_X) \subsetneq VW\mathcal{O}_X$ when F is not surjective. In particular, $W\mathcal{O}_X$ is not flat over W in general.

Remark 2.14. For k a perfect field of characteristic p, the k-points of $\mathbb{W}$ are of particular interest:

$$W = \mathbb{W}(k) = \varprojlim_n \mathbb{W}_n(k)$$

is a complete discrete valuation ring with residue field k and uniformizer $p = (0, 1, 0, 0, \dots)$. There is an augmentation homomorphism given by the canonical projection $W \to W_1 = k$. These properties characterise the ring W up to unique isomorphism inducing the identity on k. $K = W[1/p] = \operatorname{Frac} W$ is of characteristic 0.

2.3 The de Rham–Witt complex

We follow [Ill79]; for a modern treatment, via Dieudonné complexes and the saturated de Rham–Witt complex, see [BLM21].

If X is a smooth k-scheme, one could naively try to take the de Rham complex of $W(X) = (X, W\mathcal{O}_X)$ (or of $W_n\mathcal{O}_X$) and compute hypercohomology. $\rightsquigarrow$ This does not work:

- the complex is far too big: it surjects onto the correct object $W_n\Omega^\bullet_X$ (Proposition 2.18), with a large kernel;
- F and V do not extend to it with the necessary compatibilities (e.g. $FdV = d$ would require dividing by p).

Illusie's idea:

- extend to a projective system of complexes;
- extend F and V in a compatible way.

The classical picture

Definition 2.15. A graded A-algebra $B = \bigoplus_{i \geq 0} B^i$ in non-negative degrees that is *strictly anticommutative*, i.e. $xy = (-1)^{ij} yx$ for $x \in B^i$, $y \in B^j$ and $x^2 = 0$ for x of odd degree, together with an A-linear differential $d\colon B^i \to B^{i+1}$ such that

$$d^2 = 0, \qquad d(xy) = (dx)y + (-1)^i x(dy) \quad (x \in B^i),$$

is called a *differential graded algebra* (DGA). A morphism of DGAs is a morphism of graded A-algebras compatible with the differentials. Denote by $\mathrm{dga}^{\geq 0}(A)$ the category of DGAs. (Strict anticommutativity only matters when 2 is not invertible, e.g. for $p = 2$.)

Proposition 2.16. *The functor*

$$\underline{\mathrm{Alg}}(A) \longrightarrow \mathrm{dga}^{\geq 0}(A), \qquad C \longmapsto \Omega^\bullet_{C/A}$$

is left adjoint to the forgetful functor

$$\mathrm{dga}^{\geq 0}(A) \longrightarrow \underline{\mathrm{Alg}}(A), \qquad B \longmapsto B^0.$$

Illusie's picture

Definition 2.17 ([Ill79])**.** A *de Rham V-procomplex* on X consists of a projective system of DGAs (over $\mathbb{Z}$) on X

$$M_\bullet = \big((M_n)_{n\in\mathbb{Z}},\ R\colon M_{n+1} \to M_n\big)$$

and a family of additive maps $\big(V\colon M_n^i \to M_{n+1}^i\big)_{n\in\mathbb{Z}}$ such that $RV = VR$, satisfying the following properties:

- $M_n = 0$ for $n \leq 0$, M_1^0 is an $\mathbb{F}_p$-algebra, and $M_n^0 = W_n(M_1^0)$, with R, V the usual restriction and Verschiebung;
- for all $x \in M_n^i$, $y \in M_n^j$: $V(x\,dy) = (Vx)\,dVy$;
- for all $x \in M_1^0$, $y \in M_n^j$: $(Vy)\,d[x] = V\big([x]^{p-1}y\big)\,dV[x]$.

A morphism of de Rham V-procomplexes is a morphism of projective systems of DGAs compatible with V. Denote the category by $\underline{\mathrm{VDR}}(X)$.

Proposition 2.18 ([Ill79])**.** *The forgetful functor*

$$\underline{\mathrm{VDR}}(X) \longrightarrow \underline{\mathrm{Alg}}(X), \qquad M_\bullet \longmapsto M_1^0$$

from $\underline{\mathrm{VDR}}(X)$ to sheaves of $\mathbb{F}_p$-algebras on X has a left adjoint

$$\underline{\mathrm{Alg}}(X) \longrightarrow \underline{\mathrm{VDR}}(X), \qquad A \longmapsto W_\bullet\Omega^\bullet_A,$$

called the de Rham–Witt complex*. Furthermore, the natural morphism of DGAs*

$$\pi_n\colon \Omega^\bullet_{W_n(A)} \longrightarrow W_n\Omega^\bullet_A, \qquad \pi_n^0 = \mathrm{id},$$

is surjective, and $\pi_1\colon \Omega^\bullet_A \to W_1\Omega^\bullet_A$ is an isomorphism.

Proposition 2.19 ([Ill79])**.** *There exists a unique additive morphism*

$$F\colon W_n\Omega^\bullet_X \longrightarrow W_{n-1}\Omega^\bullet_X$$

satisfying

$$F(ab) = F(a)F(b), \qquad FdV = d, \qquad F(d[x]) = [x]^{p-1}d[x] \quad (x \in \mathcal{O}_X),$$

and such that $F = R \circ \mathrm{Frob}$ on $W_n\mathcal{O}_X$. Furthermore, we have the identities

$$FV = VF = p, \qquad x\,Vy = V(Fx \cdot y), \qquad dF = pFd, \qquad Vd = pdV.$$

Theorem 2.20 (Illusie [Ill79])**.** *Let X be a smooth k-scheme. There exists a canonical isomorphism between crystalline cohomology and the hypercohomology of the de Rham–Witt complex,*

$$H^\bullet(X/W_n) \cong \mathbb{H}^\bullet(X_{\mathrm{Zar}}, W_n\Omega^\bullet_X), \qquad H^\bullet(X/W) \cong \mathbb{H}^\bullet(X_{\mathrm{Zar}}, W\Omega^\bullet_X),$$

where $W\Omega^\bullet_X = \varprojlim_n W_n\Omega^\bullet_X$. Under this isomorphism the (σ-semilinear) crystalline Frobenius F^* corresponds to the endomorphism of $W\Omega^\bullet_X$ given by p^iF in degree i.

More generally, we have an isomorphism of derived objects

$$Ru_{X/W_n*}\mathcal{O}_{X/W_n} \xrightarrow{\sim} W_n\Omega^\bullet_X$$

in the derived category $D(X_{\mathrm{Zar}}, W_n)$ of sheaves of W_n-modules on X. This isomorphism is functorial in X. In particular $W_n\Omega^\bullet_X$ computes crystalline cohomology even when X has no lift to W_n.

References

[Ber74] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique $p > 0$*, Lecture Notes in Mathematics 407, Springer-Verlag, Berlin, 1974.

[BO78] P. Berthelot and A. Ogus, *Notes on Crystalline Cohomology*, Mathematical Notes 21, Princeton University Press, Princeton, 1978.

[BLM21] B. Bhatt, J. Lurie and A. Mathew, *Revisiting the de Rham–Witt complex*, Astérisque 424, Société Mathématique de France, 2021.

[CL98] A. Chambert-Loir, *Cohomologie cristalline : un survol*, Expositiones Mathematicae 16 (1998), 333–382.

[Gro68] A. Grothendieck, *Crystals and the de Rham cohomology of schemes*, in: *Dix exposés sur la cohomologie des schémas*, Advanced Studies in Pure Mathematics 3, North-Holland, Amsterdam, 1968, 306–358.

[Ill79] L. Illusie, *Complexe de de Rham–Witt et cohomologie cristalline*, Annales Scientifiques de l'École Normale Supérieure (4) 12 (1979), 501–661.

[Rob65] N. Roby, *Les algèbres à puissances divisées*, Bulletin des Sciences Mathématiques (2) 89 (1965), 75–91.