On the Geometry of Modular Curves

Notes for the study group on the Gross–Zagier formula, Leiden, Spring 2026

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Abstract

These notes accompany the second talk of the study group on the paper *Heegner points and derivatives of L-series* by Gross and Zagier. The plan is as follows.

- **Complex picture:** modular curves as quotients of the upper half plane, and the universal elliptic curve above them.
- **Moduli-theoretic picture:** moduli problems and representability; why $X(1)$ is not representable; moduli problems with level structure; generalised elliptic curves and compactifications; the reduction of $X_0(p)$ over $\mathbb{F}_p$.
- **q-expansions:** the classical description over $\mathbb{C}$ and the Tate curve.
- **Hecke operators:** as correspondences.

For the complex picture and Hecke operators we follow [DS05]; for the moduli-theoretic picture, compactifications and q-expansions we follow [DI95], [KM85] and [DR73].

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1 The classical complex picture

We follow [DS05, Ch. 1–2]. Let $\mathcal{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ be the upper half plane and $Y = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. The group $\mathrm{PSL}_2(\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad S^2 = T^3 = \mathrm{id} \text{ in } \mathrm{PSL}_2(\mathbb{Z}).$$

These are the torsion elements; they fix the *elliptic points* i and $\rho = e^{2\pi i/3}$ respectively. (Here T is S times the translation $\tau \mapsto \tau + 1$.)

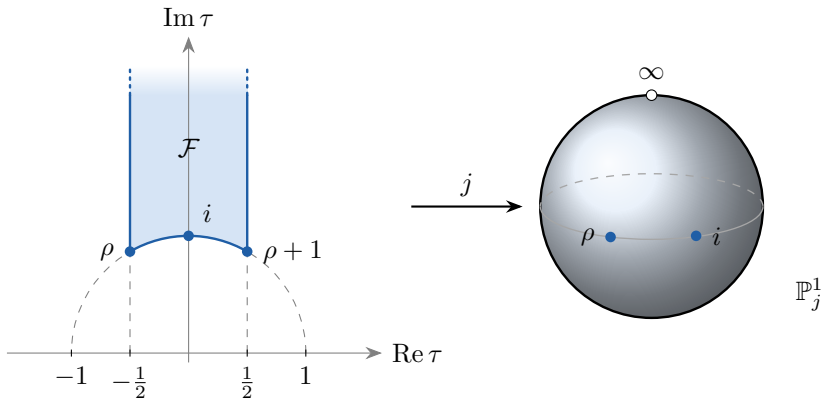


Figure 1: The standard fundamental domain $\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathcal{H}$. The j-invariant identifies $Y(1)$ with $\mathbb{A}^1$ and its compactification $X(1)$ with $\mathbb{P}^1_j$, adding the cusp ∞. The elliptic points map to $j(\rho) = 0$ and $j(i) = 1728$.

More generally, let $\Gamma(N) \leq \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and form

$$Y(\Gamma) = \Gamma\backslash\mathcal{H}.$$

A compactification is given by considering the action of Γ on $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$: $X(\Gamma) = \Gamma\backslash\big(\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})\big)$.

We want to interpret these spaces as *moduli spaces*. We are especially interested in the curves associated to

$$\Gamma_0(N) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\right\},$$
$$\Gamma_1(N) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N\right\}.$$

- $Y_0(N) = \Gamma_0(N)\backslash\mathcal{H}$. Its points are in bijection with isomorphism classes of pairs (E, C), with E an elliptic curve over $\mathbb{C}$ and $C \subset E$ a cyclic subgroup of order N. To $\tau \in Y_0(N)$ we associate
$$(E_\tau, C_\tau) = \Big(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),\ \tfrac{1}{N}\mathbb{Z}/\mathbb{Z}\Big).$$
- $Y_1(N) = \Gamma_1(N)\backslash\mathcal{H}$. Its points are in bijection with isomorphism classes of pairs (E, P), with E an elliptic curve over $\mathbb{C}$ and $P \in E$ a point of order N. To $\tau \in Y_1(N)$ we associate
$$(E_\tau, P_\tau) = \Big(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),\ \tfrac{1}{N} \bmod \mathbb{Z} + \tau\mathbb{Z}\Big).$$

Remark 1.1. For $N = 1$, $Y(1)$ is in bijection with isomorphism classes of elliptic curves over $\mathbb{C}$, and the j-invariant establishes a bijection $Y(1) \xrightarrow{\sim} \mathbb{C}$ (Figure 1).

The action of $\Gamma_0(N)$ on $\mathcal{H}$ induces an action of $\Gamma_0(N)/\Gamma_1(N)$ on $Y_1(N)$, where

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod N.$$

This action has a moduli-theoretic interpretation, the *diamond operators*

$$\langle d\rangle : (E, P) \longmapsto (E, dP).$$

Indeed, the isomorphism $\mathbb{C}/(\mathbb{Z}+\gamma\tau\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, $z \mapsto (c\tau+d)z$, sends $\frac{1}{N}$ to $\frac{c\tau+d}{N} \equiv \frac{d}{N}$, since $N \mid c$. The natural projection is

$$Y_1(N) \longrightarrow Y_0(N), \qquad (E,P) \longmapsto (E,\langle P\rangle).$$

2 The moduli-theoretic picture

2.1 Moduli problems and representability

We follow [DI95] and [KM85]. Consider $\underline{\mathrm{Sch}}$, the category of schemes. A contravariant functor $F\colon \underline{\mathrm{Sch}} \to \underline{\mathrm{Set}}$ is *representable* if there exists a scheme X such that F is isomorphic to the functor

$$h^X\colon \underline{\mathrm{Sch}} \longrightarrow \underline{\mathrm{Set}}, \qquad T \longmapsto \mathrm{Hom}_{\underline{\mathrm{Sch}}}(T,X).$$

Example 2.1.

- *Global functions.* The functor $X \mapsto \Gamma(X,\mathcal{O}_X)$ is represented by $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[T]$.
- *Multiplicative group.* The functor of units, $X \mapsto \Gamma(X,\mathcal{O}_X)^\times$, is represented by $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[T,T^{-1}]$.
- *Roots of unity.* The functor $X \mapsto \{a \in \Gamma(X,\mathcal{O}_X) : a^n = 1\}$ is represented by $\mu_n = \operatorname{Spec}\mathbb{Z}[T]/(T^n-1)$.

2.2 A naive moduli space of elliptic curves

Recall that an elliptic curve E/S is a proper smooth morphism $E \to S$ whose geometric fibres are connected curves of genus 1, together with a section $O\colon S \to E$ (Figure 2).

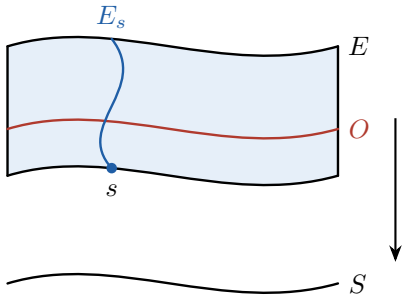


Figure 2: An elliptic curve over S: a family of genus one curves E_s, with zero section O.

Consider the functor

$$[\mathrm{Ell}]\colon \underline{\mathrm{Sch}} \longrightarrow \underline{\mathrm{Set}}, \qquad S \longmapsto \{E/S \text{ elliptic curve}\}/\cong_S .$$

This functor is not representable, because of

- the presence of automorphisms (every elliptic curve has the automorphism $[-1]$);
- bad behaviour under field extensions $\rightsquigarrow$ twists.

Suppose X represents [Ell]. Then we have a functorial isomorphism $\mathrm{Hom}_{\underline{\mathrm{Sch}}}(S,X) \cong [\mathrm{Ell}](S)$. Let L/K be a field extension. Since $\operatorname{Spec} L \to \operatorname{Spec} K$ is an epimorphism of schemes, the map

$$[\mathrm{Ell}](K) \longrightarrow [\mathrm{Ell}](L)$$

would be injective. In particular, if two elliptic curves defined over K become isomorphic over L, they would already be isomorphic over K.

Counterexample 2.2. Let $E_i\colon y^2z = x^3 + (-1)^i z^3$ for $i \in \{1, 2\}$. They are isomorphic over $\overline{\mathbb{Q}}$ via

$$E_1 \longrightarrow E_2, \qquad (x : y : z) \longmapsto (-x : iy : z),$$

but they are not isomorphic over $\mathbb{Q}$, since

$$E_1[3](\mathbb{Q}) = 0, \qquad E_2[3](\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z},$$

the latter generated by $(0 : \pm 1 : 1)$.

Counterexample 2.3. The same phenomenon happens in families, even over $\mathbb{C}$. Let $E/\mathbb{C}$ be an elliptic curve and consider two families over $\mathbb{G}_m = \operatorname{Spec}\mathbb{C}[t, t^{-1}]$:

- the trivial family $E_{\text{triv}} = E \times \mathbb{G}_m$;
- the twisted family E_{tw}: if $E\colon y^2 = f(x)$, then $E_{\text{tw}}\colon t\,y^2 = f(x)$. Analytically, it is obtained by gluing the constant family along the twist $[-1]$ as one goes once around the origin (Figure 3).

Fibrewise the curves are indistinguishable: $E_{\text{triv},t} \cong E \cong E_{\text{tw},t}$ for all $t \in \mathbb{C}^\times$, via $(x, y) \mapsto (x, \sqrt{t}\,y)$. Hence both families would correspond to the *same* morphism $\mathbb{G}_m \to X$, namely the constant map to the point $[E]$. But $E_{\text{tw}} \not\cong E_{\text{triv}}$ over $\mathbb{G}_m$, since $\sqrt{t}$ is not a function on $\mathbb{G}_m$: the family only becomes trivial after the double cover $t = u^2$. (A base such as $\mathbb{P}^1$ would not work here: since $\mathbb{P}^1$ is simply connected, every isotrivial family obtained by gluing constant families along a constant automorphism is trivial.)

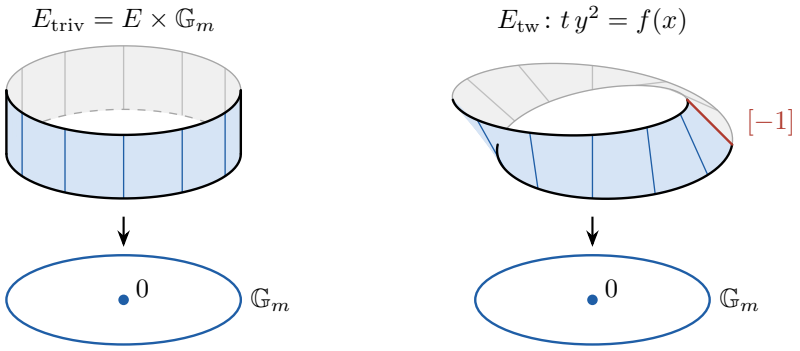


Figure 3: Schematic picture of Counterexample 2.3 over a loop around 0 in $\mathbb{G}_m(\mathbb{C})$; each segment stands for a fibre $\cong E$. Going once around the loop, the twisted family is glued back to itself via $[-1]$, which is drawn as a half twist.

2.3 Moduli problems with extra structure

Let S be a scheme over $\operatorname{Spec}\mathbb{Z}[1/N]$ and E/S an elliptic curve, so that $E[N]$ is a finite étale group scheme over S, of rank N^2.

- A *full level N-structure* on E/S is an isomorphism of finite étale group schemes

  $$\alpha\colon (\mathbb{Z}/N\mathbb{Z})^2_S \xrightarrow{\sim} E[N].$$

  Denote by $[\Gamma(N)]$ the functor classifying pairs (E, α) up to isomorphism.

- Define $[\Gamma_1(N)](S)$ as the set of isomorphism classes of pairs (E, α) with $\alpha\colon (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow E[N]$ a monomorphism of finite étale group schemes, i.e. a section of exact order N.

- Define $[\Gamma_0(N)](S)$ as the set of isomorphism classes of pairs (E, H) with $H \subset E[N]$ a closed subgroup scheme such that for every geometric point $\bar{s} \to S$ the fibre $H_{\bar{s}}$ is cyclic of order N.

Theorem 2.4 ([KM85, DI95])**.**

- *If $N > 3$, $[\Gamma_1(N)]$ is representable by a smooth affine scheme $\mathcal{Y}_1(N)$ of relative dimension 1 over* $\operatorname{Spec} \mathbb{Z}[1/N]$.
- *If $N > 2$, $[\Gamma(N)]$ is representable by a smooth affine scheme $\mathcal{Y}(N)$ of relative dimension 1 over* $\operatorname{Spec} \mathbb{Z}[1/N]$.

2.4 Universal elliptic curves

Let $N > 3$, so that $[\Gamma_1(N)]$ is representable:

$$\begin{aligned}[\Gamma_1(N)]\colon \underline{\mathrm{Sch}}_{\mathbb{Z}[1/N]} &\longrightarrow \underline{\mathrm{Set}},\\ S &\longmapsto \{(E,\alpha) : E/S \text{ elliptic curve, } \alpha \text{ a } \Gamma_1(N)\text{-structure}\}/\cong_S\\ &\cong \mathrm{Hom}_{\underline{\mathrm{Sch}}_{\mathbb{Z}[1/N]}}(S, \mathcal{Y}_1(N)) = \mathcal{Y}_1(N)(S).\end{aligned}$$

$\rightsquigarrow$ The special element $\mathrm{id} \in \mathrm{Hom}(\mathcal{Y}_1(N), \mathcal{Y}_1(N))$ corresponds to a pair $(\mathcal{E}^{\mathrm{univ}}, \alpha^{\mathrm{univ}})$, where $\mathcal{E}^{\mathrm{univ}}/\mathcal{Y}_1(N)$ is an elliptic curve: the *universal elliptic curve*.

Fix a scheme S over $\mathbb{Z}[1/N]$, let $(E, \alpha) \in [\Gamma_1(N)](S)$, and let $\varphi \in \mathcal{Y}_1(N)(S)$ be the corresponding element. Then we have a cartesian diagram

$$\begin{array}{ccc}(E,\alpha) & \longrightarrow & (\mathcal{E}^{\mathrm{univ}}, \alpha^{\mathrm{univ}})\\ \downarrow & \lrcorner & \downarrow\\ S & \xrightarrow[\varphi]{} & \mathcal{Y}_1(N),\end{array}$$

i.e. $(E, \alpha) \cong \varphi^*(\mathcal{E}^{\mathrm{univ}}, \alpha^{\mathrm{univ}})$.

A concrete picture over $\mathbb{C}$

Let $\mathbb{Z}^2$ act on $\mathcal{H} \times \mathbb{C}$ by $(m, n) \cdot (\tau, z) = (\tau, z + m + n\tau)$, and $\Gamma_1(N)$ by $\gamma \cdot (\tau, z) = \left(\gamma\tau, \frac{z}{c\tau+d}\right)$. For $N > 3$ the group $\Gamma_1(N)$ is torsion free and acts freely on $\mathcal{H}$, and

$$(\mathcal{E}^{\mathrm{univ}})^{\mathrm{an}} \cong \Gamma_1(N)\backslash\left(\mathcal{H} \times \mathbb{C}/\mathbb{Z}^2\right) \longrightarrow \mathcal{Y}_1(N)(\mathbb{C})^{\mathrm{an}} \cong \Gamma_1(N)\backslash\mathcal{H},$$

with fibre over τ the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and the universal point $\frac{1}{N}$ (Figure 4).

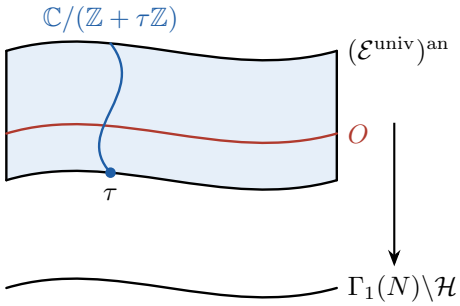


Figure 4: The universal elliptic curve over $Y_1(N)(\mathbb{C})$.

An algebraic example

$[\Gamma_1(4)]$ is representable by

$$\mathcal{Y}_1(4) = \operatorname{Spec} \mathbb{Z}\Big[\tfrac{1}{2},\, d,\, \big(d(d-4)\big)^{-1}\Big].$$

The universal elliptic curve is given by

$$d\,y^2z = x^3 + (d-2)\,x^2z + xz^2,$$

and the universal section of order 4 is $P = (1:1:1)$. (The tangent line at P is $y = x$, which meets the curve again at $(0:0:1)$, a point of order 2; hence $2P = (0:0:1)$ and P has exact order 4.)

2.5 The case $[\Gamma_0(N)]$

$[\Gamma_0(N)]$ classifies pairs (E, H) with $H \subset E[N]$ a locally free cyclic subgroup scheme of rank N. The automorphism $[-1]$, which preserves every H, prevents representability.

Let $N > 3$ and consider $(\mathcal{E}^{\text{univ}}, P^{\text{univ}})$, the universal elliptic curve over $\mathcal{Y}_1(N)$. For $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ the pair $(\mathcal{E}^{\text{univ}}, dP^{\text{univ}})$ defines a morphism

$$\langle d\rangle \colon \mathcal{Y}_1(N) \longrightarrow \mathcal{Y}_1(N) \qquad \rightsquigarrow \qquad (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \operatorname{Aut}(\mathcal{Y}_1(N)).$$

Since $[-1]$ is an isomorphism $(E, P) \cong (E, -P)$, the element $\langle -1\rangle$ acts trivially, so the action factors through $G = (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}$. Consider the quotient scheme

$$\mathcal{Y}_0(N) \coloneqq G\backslash\mathcal{Y}_1(N).$$

This defines the *coarse moduli scheme* for $[\Gamma_0(N)]$ (see [KM85, DI95]). $\rightsquigarrow$ For k an algebraically closed field of characteristic prime to N, $\mathcal{Y}_0(N)(k)$ can be identified with the set of isomorphism classes of pairs (E, H) over k. Over a general field k, two pairs defined over k give the same k-point if and only if they become isomorphic over $\bar{k}$.

The quotient map $\mathcal{Y}_1(N) \to \mathcal{Y}_0(N)$ is finite flat, but not necessarily étale: it ramifies at the pairs with extra automorphisms.

2.6 Bad reduction of $Y_0(p)$

Let $p \nmid N$ and consider the moduli problem $[\Gamma_1(N, p)]$ over $\mathbb{Z}[1/N]$-schemes classifying triples (E, α, C) over S, where

- E/S is an elliptic curve;
- $\alpha \colon (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow E[N]$ is a monomorphism of group schemes;
- $C \subset E$ is a finite locally free subgroup scheme of rank p.

Theorem 2.5 (Deligne–Rapoport [DR73]; see also [KM85, DI95])**.** *If $N > 3$, then $[\Gamma_1(N, p)]$ is representable by a scheme $\mathcal{Y} = \mathcal{Y}_1(N, p)$ over $\mathbb{Z}[1/N]$. Moreover, $\mathcal{Y}$ is regular, and $\mathcal{Y}_{\mathbb{Z}[1/Np]}$ is smooth with irreducible geometric fibres.*

Consider

$$\mathcal{Y}_{\mathbb{F}_p} = \mathcal{Y} \times \mathbb{F}_p, \qquad \mathcal{Y}_1(N)_{\mathbb{F}_p} = \mathcal{Y}_1(N) \times \mathbb{F}_p, \qquad \mathcal{E}_0 = (\mathcal{E}^{\text{univ}})_{\mathbb{F}_p}, \qquad \alpha_0 = (\alpha^{\text{univ}})_{\mathbb{F}_p}.$$

- The Frobenius induces a p-isogeny $F \colon (\mathcal{E}_0, \alpha_0) \to (\mathcal{E}_0^{(p)}, \alpha_0^{(p)})$. This gives a closed immersion
$$i_F \colon \mathcal{Y}_1(N)_{\mathbb{F}_p} \longrightarrow \mathcal{Y}_{\mathbb{F}_p}, \qquad (E, \alpha) \longmapsto (E, \alpha, \ker F).$$

- The Verschiebung induces a p-isogeny $V\colon (\mathcal{E}_0^{(p)}, d\,\alpha_0^{(p)}) \to (\mathcal{E}_0, \alpha_0)$, with $dp \equiv 1 \bmod N$ (because $V \circ F = p$). This gives a closed immersion

$$i_V\colon \mathcal{Y}_1(N)_{\mathbb{F}_p} \longrightarrow \mathcal{Y}_{\mathbb{F}_p}, \qquad (E, \alpha) \longmapsto \big(E^{(p)}, d\,\alpha^{(p)}, \ker V\big).$$

The special fibre $\mathcal{Y}_{\mathbb{F}_p}$ is the union of the images of i_F and i_V, two copies of $\mathcal{Y}_1(N)_{\mathbb{F}_p}$, which cross transversally at the supersingular points (Figure 5).

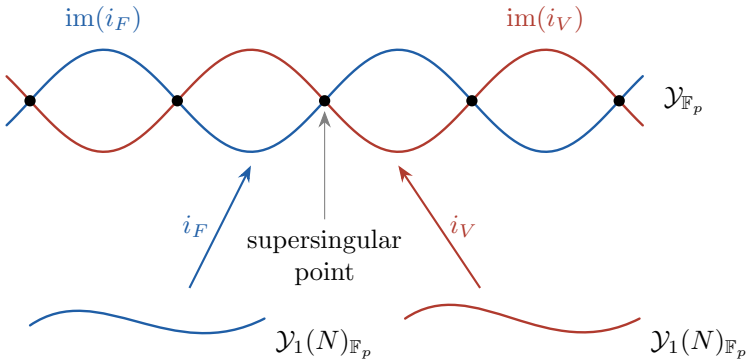


Figure 5: The special fibre of $\mathcal{Y}_1(N, p)$ at p: two copies of $\mathcal{Y}_1(N)_{\mathbb{F}_p}$, embedded via Frobenius and Verschiebung, crossing at the supersingular points.

3 Compactification

Over $\mathbb{C}$ we have the open immersion

$$Y(\Gamma) \hookrightarrow \Gamma\backslash\big(\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})\big) = X(\Gamma),$$

i.e. we add finitely many points, the *cusps*, to compactify the modular curve. Deligne and Rapoport [DR73] introduced a moduli description of the cusps (see also [DI95]) $\rightsquigarrow$ *degenerations of elliptic curves.*

3.1 Néron polygons and generalised elliptic curves

Definition 3.1. Let k be an algebraically closed field. The *Néron N-gon* $(C_N, +)$ is the scheme obtained by gluing N copies of $\mathbb{P}^1$,

$$\nu\colon \coprod_{i \in \mathbb{Z}/N\mathbb{Z}} \mathbb{P}^1 \longrightarrow C_N,$$

where $(\infty)_i$ and $(0)_{i+1}$ are sent to the same point. The map ν restricts to an isomorphism on the smooth locus,

$$\nu\colon \coprod_{i \in \mathbb{Z}/N\mathbb{Z}} \mathbb{G}_m \xrightarrow{\sim} C_N^{\mathrm{sm}} \cong \mathbb{G}_m \times \mathbb{Z}/N\mathbb{Z},$$

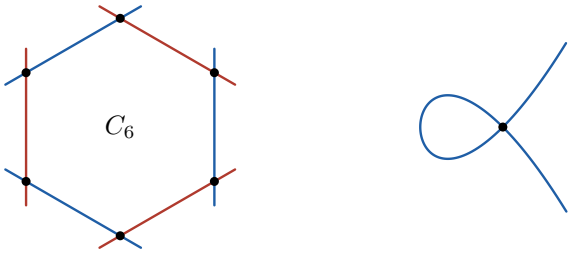
inducing a group structure on C_N^{sm} and a natural action

$$C_N^{\mathrm{sm}} \times C_N \longrightarrow C_N, \qquad \big((x)_i, (y)_j\big) \longmapsto (xy)_{i+j}.$$

Definition 3.2. A *generalised elliptic curve* over S is a pair $(E, +)$ with

- $E \to S$ a proper, flat curve of finite presentation;
- $+\colon E^{\mathrm{sm}} \times_S E \to E$ an S-morphism which restricts to a commutative group law on the smooth locus E^{sm} and defines an action of E^{sm} on E,

such that the geometric fibres of $(E, +)$ are either elliptic curves or Néron polygons $(C_N, +)$.



Néron 6-gon Néron 1-gon

Figure 6: Néron polygons: each side is a copy of $\mathbb{P}^1$, and ∞ on one side is glued to 0 on the next. The 1-gon is the nodal cubic.

3.2 The Tate curve

Example 3.3. The *Tate curve* over $S = \operatorname{Spec} \mathbb{Z}[[q]]$ is defined by

$$E_q \colon Y^2Z + XYZ = X^3 + a_4XZ^2 + a_6Z^3 \quad \text{in } \mathbb{P}^2_S,$$

with

$$a_4 = -5\sum_{n\geq 1} \frac{n^3q^n}{1-q^n}, \qquad a_6 = -\frac{1}{12}\sum_{n\geq 1} \frac{(7n^5+5n^3)q^n}{1-q^n}.$$

(The coefficients of a_6 are integers.) It is a generalised elliptic curve. Its smooth locus E_q^{sm} is the complement of the closed subscheme $X = Y = q = 0$.

- Let $s \colon \mathbb{Z}[[q]] \to k$ define a geometric point with $s(q) = 0$. Then

  $$E_{q,s} \colon Y^2Z + XYZ = X^3$$

  is the Néron 1-gon (Figure 6).

- The series a_4, a_6 converge for $|q| < 1$. Specialising $q = e^{2\pi i\tau}$ with $\tau \in \mathcal{H}$ gives an elliptic curve over $\mathbb{C}$ with a canonical uniformisation

  $$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \xrightarrow[\sim]{z\mapsto e^{2\pi iz}} \mathbb{C}^\times/q^{\mathbb{Z}} \xrightarrow{\sim} E_q(\mathbb{C}) \qquad \text{(Figure 7).}$$

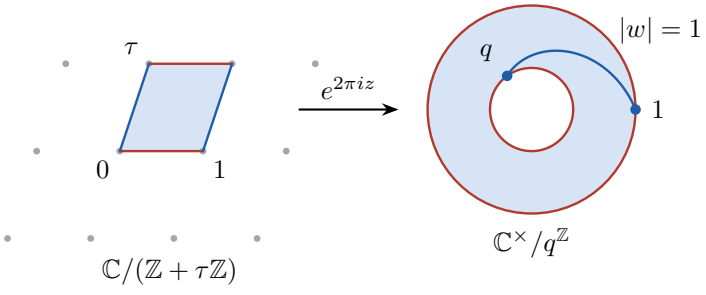


Figure 7: Uniformisation of the Tate curve over $\mathbb{C}$ (here $\tau = 0.35 + 1.05\,i$). The segment $[0,1]$ wraps around the unit circle, $[0,\tau]$ becomes a spiral from 1 to $q = e^{2\pi i\tau}$, and the two boundary circles $|w| = 1$ and $|w| = |q|$ of the annulus are identified by $w \sim qw$ (radii not to scale).

3.3 Compactified moduli problems and q-expansions

Consider the functor

$$\mathcal{G}_1(N)\colon \underline{\mathrm{Sch}}_{\mathbb{Z}[1/N]} \longrightarrow \underline{\mathrm{Set}}$$

associating to a scheme S the set of isomorphism classes of pairs (E, P), where

- E is a generalised elliptic curve over S;
- $P\colon S \to E^{\mathrm{sm}}$ is a section of exact order N, such that for every geometric point $s\colon \operatorname{Spec} k \to S$ the image of the resulting immersion $(\mathbb{Z}/N\mathbb{Z})_k \hookrightarrow E_s^{\mathrm{sm}}$ meets every irreducible component of E_s.

Theorem 3.4 (Deligne–Rapoport [DR73])**.** *If $N > 4$, then $\mathcal{G}_1(N)$ is representable by a proper smooth curve $\mathcal{X}_1(N)$ over $\operatorname{Spec}\mathbb{Z}[1/N]$, containing $\mathcal{Y}_1(N)$ as the complement of the cusps.*

One can define an analogous problem for $\Gamma_0(N)$, again with a coarse moduli scheme $\mathcal{X}_0(N)$.

The Tate curve has no point of exact order N defined over $\mathbb{Z}[1/N][[q]]$ when $N > 2$, but over $R = \mathbb{Z}[1/N, \zeta_N]$ the root of unity $\zeta_N \in \mu_N \subset \mathbb{G}_m \subset E_q^{\mathrm{sm}}$ is such a point. Over $\mathbb{C}$, it corresponds to $P_\tau = \frac{1}{N}$ under the uniformisation of Figure 7. We then obtain a canonical map

$$\begin{array}{ccc} E_q & \longrightarrow & \mathcal{E}^{\mathrm{univ}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R[[q]] & \xrightarrow[i_q]{} & \mathcal{X}_1(N), \end{array} \qquad (E_q, \zeta_N) \longmapsto i_q^*(\mathcal{E}^{\mathrm{univ}}, P^{\mathrm{univ}}).$$

Think of q as a local parameter at the cusp ∞. (Alternatively, working with the moduli problem of embeddings $\mu_N \hookrightarrow E[N]$ instead of $(\mathbb{Z}/N\mathbb{Z}) \hookrightarrow E[N]$, the cusp ∞ is already defined over $\mathbb{Z}[1/N]$; see [DI95] for a discussion of these conventions.)

From this construction we obtain the *q-expansion map*. Let

$$\omega_f \in H^0\big(\mathcal{X}_1(N)_R,\ \Omega^1_{\mathcal{X}_1(N)/R}(\mathrm{cusps})\big),$$

i.e. a differential on $\mathcal{Y}_1(N)_R$ with at most simple poles at the cusps. Then

$$i_q^*\omega_f = f(q)\,\frac{dq}{q}, \qquad f(q) \in R[[q]],$$

and ω_f is regular at ∞ (a cusp form) if and only if $f(0) = 0$. For ω_f only defined on $\mathcal{Y}_1(N)$, restricting along $\operatorname{Spec} R((q)) \to \mathcal{Y}_1(N)$ gives $f(q) \in R((q))$.

4 Hecke operators

We interpret the Hecke operators in terms of correspondences, following [DS05, Ch. 5 and 7] and [DI95]. Let $p \nmid N$:

$$\begin{array}{ccccc} & & \mathcal{Y}_1(N,p) & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ \mathcal{Y}_1(N) & & & & \mathcal{Y}_1(N) \end{array} \qquad \begin{array}{ccc} & (E,\alpha,C) & \\ \swarrow & & \searrow \\ (E,\alpha) & & (E/C,\alpha') \end{array}$$

where $\alpha' = \pi_C \circ \alpha\colon (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow (E/C)[N]$ and $\pi_C\colon E \to E/C$ is the quotient isogeny. Since $p \nmid N$, π_C is an isomorphism on N-torsion, so α' is again a $\Gamma_1(N)$-structure.

The action on modular forms is induced by pullback and pushforward under these maps. On weight 2 forms, viewed as differentials, the Hecke operator is

$$T_p = \pi_{2*} \circ \pi_1^*,$$

where π_{2*} is the trace along the finite flat map π_2 (up to the choice of normalisation conventions).

References

[DI95] F. Diamond and J. Im, *Modular forms and modular curves*, in: *Seminar on Fermat's Last Theorem (Toronto, 1993–1994)*, CMS Conference Proceedings 17, American Mathematical Society, Providence, RI, 1995, 39–133.

[DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, in: *Modular Functions of One Variable II*, Lecture Notes in Mathematics 349, Springer-Verlag, Berlin, 1973, 143–316.

[DS05] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics 228, Springer-Verlag, New York, 2005.

[KM85] N. M. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies 108, Princeton University Press, Princeton, 1985.