ALGANT Master thesis in Mathematics

Two-variable p-adic L-function

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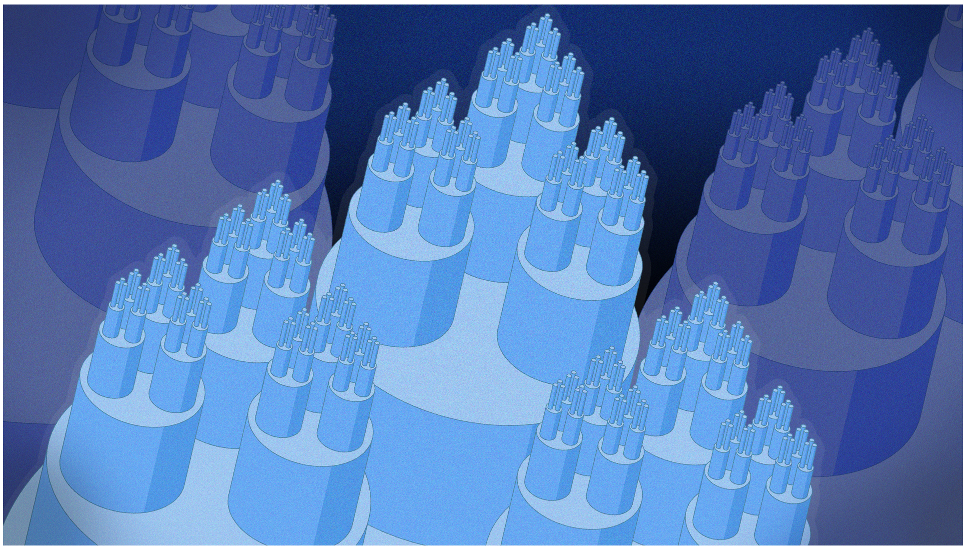


Figure 1: A visualization of the 3-adic numbers - Samuel Velasco/Quanta Magazine

CHAPTER 1

Introduction

This mémoire aims to study the construction of p-adic L-functions attached to a quadratic imaginary field with respect to a particular Hecke character. The complex L-functions are a widely studied object that arise in different branches of Mathematics. It is possible to construct p-adic measures that interpolate some special L-values in order to create a new arithmetic object that enable us to study its properties from an algebraic point of view.

T. Kubota and H.W. Leopold [LK64] gave a first construction of p-adic L-function for the Riemann ζ-function and its twists. They used Kummer's congruences of Bernoulli numbers B_n to interpolate particular negative values of the Riemann ζ function. Later, K. Iwasawa [Iwa69] discovered that the Bernoulli numbers and their properties arise from the arithmetic of towers of cyclotomic fields. Analogously to the previous construction, Manin and Vishik [VM74] and Katz [Kat76] constructed p-adic L functions which interpolates special values of Hecke L-series associated with a quadratic imaginary field K, in which p-splits. It happens that special values of Eisenstein series attached to elliptic curves with complex multiplication interpolate the values of L-functions in the same way as Bernoulli numbers do for the Riemann ζ function. Coates and Wiles described in two fundamental papers [CW77], [CW78] a norm-coherent sequence of elliptic units encoding the Eisenstein numbers properties and involving tower extensions of K with ray class fields generated by torsion points of the elliptic curve associated to K. This approach has been furtherly developed by R. Yager [Yag82] for class number 1 and E. de Shalit [DS87] for the general construction. In this paper we will closely follow Yager's construction.

More concretely, consider K a quadratic imaginary field and $\mathcal{O}_K$ its ring of integers. We will always assume that K has class number 1. We can then consider an elliptic curve E defined over K with complex multiplication by $\mathcal{O}_K$. By the theory of complex multiplication, there exists a Hecke character ψ on K attached to E that encodes the arithmetic structure of E. Let $p \neq 2, 3$ be a rational prime such that E has good reduction above it. We will assume p splits in K, $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Fix a Weierstrass model for E

$$y^2 = 4x^3 - g_2 x - g_3$$

such that $g_2, g_3 \in \mathcal{O}_K$ and its discriminant is prime to p. Let $L \subset \mathbb{C}$ the lattice associated to the model, then there exists $\Omega_\infty \in \mathbb{C}$ such that $L = \Omega_\infty \mathcal{O}_K$. For each pair $(i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$, Katz has proved the existence of a power series $\mathcal{G}^{(i_1,i_2)}$ with coefficients in the ring of integers A of a certain unramified extension of the completion of K at $\mathfrak{p}$ with the following interpolation property. If $0 \leq j < k$, we write

$$L_\infty(\overline{\psi}^{k+j}, k) = (1-\psi(\mathfrak{p})^{k+j}(N\mathfrak{p})^{-j+1})(1-\overline{\psi}(\mathfrak{p}^*)^{k+j}(N\mathfrak{p}^*)^{-k}) \left(\frac{2\pi}{\sqrt{d_K}}\right)^j \Omega_\infty^{-k-j} L(\overline{\psi}^{k+j}, k)$$

where d_K is the discriminant of K and we fix a generator u of $(1+p\,\mathbb{Z}_p)^\times$. Then for each (k_1,k_2) pair of integers satisfying $k_1 > -k_2 \geq 0$ and $(k_1,k_2) \equiv (i_1,i_2) \mod (p-1)$ we have

$$\mathcal{G}^{(i_1,i_2)}(u^{k_1}-1, u^{k_2}-1) = (k_1-1)!\Omega_{\mathfrak{p}}^{k_2-k_1} L_\infty(\overline{\psi}^{k_1-k_2}, k_1).$$

To construct such a power series we will study the properties of a rational function on E defined by

$$\Theta_{E,\mathfrak{a}} = \alpha^{-12}\Delta(E)^{N\mathfrak{a}-1} \prod_{P\in E_{\mathfrak{a}}-O} (x - x(P))^{-6}$$

and in particular of some special values at torsion points called *elliptic units*. These elements correspond to a generalization of the cyclotomic units in $\mathbb{Q}_p$ and live in ray class field extensions of K. These extensions form a tower and they are generated by the coordinate of the torsion points of the elliptic curve E. The Lubin-Tate theory in Chapter 2 describes the properties of these tower extensions. The theta function comes to play a role in the interpolation of L-functions because the Laurent expansion has coefficients of the form

$$\left(\frac{d}{dz}\right)^k \log\Theta(z,\mathfrak{a}) = (-1)^{k-1}12(k-1)!(N\mathfrak{a}E_k(z,L) - E_k(z,\mathfrak{a}^{-1}L)).$$

where E_k are the Eisenstein-Kronecker functions and they have the property

$$E_k(\rho, L) = \rho^{-k}\psi(\mathfrak{c})^k L_{\mathfrak{m}}(\overline{\psi}^k, k, \mathfrak{c})$$

with ρ an $\mathfrak{m}$-division point. With all these elements we can eventually construct a power series $\mathcal{G}^{(i_1,i_2)}$ interpolating the L function attached to K and of character $\overline{\psi}$.

In the last chapter we will study the properties of a two variables theta function $\Theta_{(z_0,w_0)}(z,w)$ that encodes the information of the Eisentsein-Kronecker series allowing us to reobtain the construction of Yager's interpolation from a different point of view. In particular, the Laurent expansion of this function is of the form

$$\Theta_{z_0,w_0}(z,w) = \langle w_0, z_0\rangle\delta_{z_0}z^{-1} + \delta_{w_0}w^{-1} + \sum_{k,j\geq 0}(-1)^{k+j}\frac{E_{j,k+1}(z_0,w_0)}{A^j j!}z^k w^j.$$

In this Mémoire we used this property to actually compute the Eisenstein numbers at different tosion points, and recognize them as algebraic numbers. All the computations have been developed with GP/PARI.

1.1 Background and notation

1.1.1 Quadratic imaginary fields

Let K quadratic imaginary field with $\mathcal{O}_K$ ring of integers, then there exists $d \in \mathbb{Q}$ such that $K = \mathbb{Q}(\sqrt{d})$. We denote by $-d_K$ the discriminant of K and we have

$$-d_K = \begin{cases} d \text{ if } d \equiv 1 \mod 4 \\ 4d \text{ otherwise.} \end{cases} \tag{1.1}$$

By *Dirichlet's unit theorem*, we have that $\mathcal{O}_K^\times = \omega_K$ where ω_K is the set of roots of unity in K. Since $K/\mathbb{Q}$ is quadratic, then $\omega_K = 2, 4$ or 6. We fix an embedding $i : \overline{Q} \hookrightarrow \mathbb{C}$.

Let $\mathfrak{f}, \mathfrak{g} \subseteq \mathcal{O}_K$ be two integral ideals, then we denote $\omega_\mathfrak{f}$ the number of roots of unity congruent to $1 \mod \mathfrak{f}$. The ray class field modulo $\mathfrak{f}$ is denoted by $K(\mathfrak{f})$ and we define $K(\mathfrak{f}\mathfrak{g}^\infty)$ by

$$K(\mathfrak{f}\mathfrak{g}^\infty) = \bigcup_{n \geq 0} K(\mathfrak{f}\mathfrak{g}^n).$$

We denote by $H = K(1)$ the Hilbert class field. By Corollary A.4.1 we have

$$[K(\mathfrak{g}) : H] = h_K^\mathfrak{g}/h_K = \frac{\omega_\mathfrak{g}}{\omega_K} \#(O_K/\mathfrak{g})^\times. \tag{1.2}$$

1.1.2 Elliptic curve associated to K

Consider K to be an imaginary quadratic field with class number 1, then K coincides with the Hilbert class field $H = K$ and $\mathcal{O}_K$ is a PID. By Corollary B.1.2, there exists an elliptic curve E defined over K with complex multiplication by $\mathcal{O}_K$. Denote by S the finite set consisting of 2,3 and the rational primes q such that E has bad reduction at least one prime above q. We fix a Weierstrass model for E

$$y^2 = 4x^3 - g_2 x - g_3 \tag{1.3}$$

where $g_2, g_3 \in \mathcal{O}_K$ and the discriminant of (1.3) is divisible only by primes of K lying above primes in S. This is possible because K has class number 1 (See [Sil94].VIII.8). Let $\wp(z)$ the Weierstrass function associated with (1.3) and L the period lattice of $\wp(z)$. As usual, we have an analytic morphism

$$\xi : \mathbb{C}/L \to E(\mathbb{C})$$

with $\xi(z) = (\wp(z), \wp'(z))$. We can identify $\mathcal{O}_K$ with the endomorphism ring in such a way that the endomorphism corresponding to $\alpha \in \mathcal{O}_K$ is given by $\xi(z) \mapsto \xi(\alpha z)$. Choose an element Ω_∞ of the period lattice L such that

$$L = \Omega_\infty \mathcal{O}_K.$$

Let ψ be the Hecke character associated to E over K defined by Theorem B.3 and fix a generator f of $\mathfrak{f} = (f)$. We fix a prime $\mathfrak{p}$ of K lying above a rational prime p such that $p \notin S$ and $\mathfrak{p}$ is of degree 1. Hence $\mathfrak{p}$ has good reduction, it is coprime to $6\mathfrak{f}$ and we will write

$$(p) = \mathfrak{p}\overline{\mathfrak{p}}.$$

Put $\pi = \psi(\mathfrak{p})$ and $\overline{\pi} = \psi(\overline{\mathfrak{p}})$, and observe that from Corollary B.3.1.(i) they are generators of the respective ideals

$$\mathfrak{p} = (\pi), \quad \overline{\mathfrak{p}} = (\overline{\pi}).$$

We will denote E_α to be the kernel of the endomorphism α of E and let

$$E_{\pi^\infty} = \bigcup_{n \geq 0} E_{\pi^{n+1}}, \quad E_{\overline{\pi}^\infty} = \bigcup_{m \geq 0} E_{\overline{\pi}^{m+1}}.$$

CHAPTER 2

Elliptic tower field

Cyclotomic extensions of $\mathbb{Q}_p$ have been broadly studied and their properties have been remarkably important for the ideas behind Class Field Theory and p-adic interpolation of ζ-function. In this chapter, we will study how the properties of this tower field extension can come from a more general construction called Lubin-Tate groups. In particular, we will use this theory to study the extension of the quadratic imaginary field K with torsion points of the elliptic curve E attached to it. These towers will be the setting in which the theory of the elliptic units develops.

2.1 Formal groups

First of all, we recall the basic properties of the formal groups and their formalism. The formal groups can give the structure of a group to complete algebras over a fixed ring, this allows us to study the properties of the power series instead of the single group.

2.1.1 Definitions and first properties

Let R be a commutative ring with identity.

Definition 2.1. *A formal group $\mathcal{F}$ defined over R is a power series $F(X,Y) \in R[[X,Y]]$ satisfying:*

(i) $F(X,Y) = X + Y +$*(terms of degree* ≥ 2*);*

(ii) $F(X, F(Y,Z)) = F(F(X,Y), Z)$ *(associativity);*

(iii) $F(X,Y) = F(Y,X)$ *(commutativity);*

(iv) *there is a unique power series* $\iota(T) \in R[[T]]$ *such that* $F(T, \iota(T)) = 0$ *(inverse);*

(v) $F(X,0) = X$ *and* $F(0,Y) = Y$*.*

We call $F(X,Y)$ the formal group law of $\mathcal{F}$.

Definition 2.2. *The formal additive group, denoted $\hat{\mathbb{G}}_a$ is given by*

$$F(X,Y) = X + Y.$$

The formal multiplicative group, denoted $\hat{\mathbb{G}}_m$ is given by

$$F(X,Y) = X + Y + XY = (1+X)(1+Y) - 1.$$

Let A be an R-algebra and $\mathfrak{a}$ an ideal such that A is complete and separated in its $\mathfrak{a}$-adic topology. Then if $f, g \in \mathfrak{a}$, $F(f,g)$ and $\iota(f)$ converge to elements of $\mathfrak{a}$, denoted respectively by $f[+]g$ and $[-]f$. Observe that with $[+]$ as addition $\mathfrak{a}$ becomes an abelian group, we write $F(\mathfrak{a})$ to distinguish it from the ordinary addition on $\mathfrak{a}$.

Definition 2.3. *Let $(\mathcal{F}, F)$ and $(\mathcal{G}, G)$ be formal power groups defined over R. A homomorphism from $\mathcal{F}$ to $\mathcal{G}$ defined over R is a power series with no constant term $f(T) \in R[[T]]$ satisfying*

$$f(F(X,Y)) = G(f(X), f(Y)).$$

$\mathcal{F}$ and $\mathcal{G}$ are isomorphic over R if there are homomorphism $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{F}$ defined over R with

$$f(g(T)) = g(f(T)) = T.$$

The collection $Hom(F,G)$ of such homomorphism forms a group with respect to the addition law of G

$$(f[+]g)(T) := G(f(T), g(T)),$$

and $End(F)$ becomes a ring under composition as a product.

Theorem 2.1. *Let R be a domain of characteristic 0, and $f \in Hom(F, F')$. Then $F(T) = aT+$(higher terms) induces an injective group homomorphism*

$$\begin{aligned} Hom(F, F') &\to R \\ f &\mapsto f'(0) = a. \end{aligned}$$

Proof. See [Haz78] 20.1. ☐

For $a \in R$ we denote $[a]_{F,F'} \in Hom(F, F')$ and $[a]_F \in End(F)$ the unique elements such that $[a]'_{F,F'}(0) = [a]'_F(0) = a$.

Definition 2.4. *An invariant differential on $\mathcal{F}/R$ is a differential form ω*

$$\omega(T) = P(T)dT \in R[[T]]dT$$

satisfying

$$\omega \circ F(T,S) = \omega(T).$$

In other words, satisfying

$$P(F(T,S))F_X(T,S) = P(T),$$

where $F_X(T,S)$ is the partial derivative of F with respect to the first variable. An invariant differential as above is said to be normalized if $P(0) = 1$.

Prop. 2.1.1. *Let $\mathcal{F}/R$ be a formal group. There exists a unique normalized invariant differential on $\mathcal{F}/R$, given by the formula*

$$\omega = F_X(0,T)^{-1}dT.$$

Every invariant differential on $\mathcal{F}/R$ is of the form $a\omega$ for some $a \in R$.

Proof. See [Sil09] 4.2 □

Definition 2.5. *Let R be a ring of characteristic 0, $K = R \otimes \mathbb{Q}$, and $\mathcal{F}/R$ be a formal group. Let*

$$\omega(T) = (1 + c_1T + c_2T^2 + c_3T^3 + \cdots)dT$$

be the normalized invariant differential on $\mathcal{F}/R$. The formal logarithm $\lambda_{\mathcal{F}} \in K[[T]]$ of $\mathcal{F}/R$ is the power series

$$\lambda_{\mathcal{F}}(T) = \int \omega = T + \frac{c_1}{2}T^2 + \frac{c_2}{3}T^3 + \cdots \in K[[T]].$$

The formal exponential of $\mathcal{F}/R$ is the unique power series $\varepsilon_{\mathcal{F}}(T) \in K[[T]]$ satisfying

$$\lambda_{\mathcal{F}}(T) \circ \varepsilon_{\mathcal{F}}(T) = \varepsilon_{\mathcal{F}}(T) \circ \lambda_{\mathcal{F}}(T) = T.$$

Observe that by definition we have that $\lambda'(T) \in R[[T]]^\times$ has coefficients in R.

Prop. 2.1.2. *Let $\mathcal{F}/R$ be a formal group with $char(R) = 0$. Then*

$$\lambda_{\mathcal{F}} : \mathcal{F} \to \hat{\mathbb{G}}_a$$

is an isomorphism of formal groups over $K = R \otimes \mathbb{Q}$ with inverse $\varepsilon_{\mathcal{F}}$.

Proof. See [Sil09] 5.2. □

We conclude with the definition of height for rings of finite characteristic.

Definition 2.6. *Let F be a formal group over a field of characteristic $p > 0$. Then $[p]_F(T) = T[+]\cdots[+]T$ (p times) is a power series in X^q with $q = p^h$ for some $h > 0$. The largest possible h is called the height of F. If $[p]_F = 0$, then F is said to be of infinite height.*

2.1.2 Lubin-Tate groups

Let k be a finite extension of $\mathbb{Q}_p$, let $\mathcal{O}$ and $\mathfrak{p}$ be its valuation ring and maximal ideal. Let the residue field $\mathcal{O}/\mathfrak{p}$ have q elements. Lubin-Tate [LT65] introduced an extremely useful class of formal groups defined over $\mathcal{O}$ that possess a special endomorphism that lifts the Frobenius substitution $X \mapsto X^q$.

Definition 2.7. *Let $\mathcal{F}_\pi$ denote the set of power series $f(T) \in \mathcal{O}[[T]]$ which satisfy the two conditions*

(i) $f(T) \equiv \pi T \mod \deg 2$;

(ii) $f(T) \equiv T^q \mod \mathfrak{p}$;

where π is a uniformizer of $\mathcal{O}$. The simplest choice for an element $f \in \mathcal{F}_\pi$ is $f(T) = \pi T + T^q$.

Lemma 2.1.1. *Let $f(T), g(T)$ be elements of $\mathcal{F}_\pi$, and let $L(X_1, \dots, X_n) = \sum_{i=1}^n a_i X_i$ be a linear form with coefficients in $\mathcal{O}$. Then there exists a unique series $F(X_1, \dots, X_n) \in \mathcal{O}[[X_1, \dots, X_n]]$ such that*

(i) $F(X_1, \dots, X_n) \equiv L(X_1, \dots, X_n) \mod \deg 2$,

(ii) $f(F(X_1, \dots, X_n)) = F(g(X_1, \dots, g(X_n)))$.

Proof. See [LT65]. □

Definition 2.8. *For each $f \in \mathcal{F}_\pi$, we let $F_f(X, Y)$ be the series associated to the linear form $L = X + Y$ in Lemma 2.1.1. In particular, F_f satisfies the following properties*

(i) $F_f(X, Y) \equiv X + Y \mod \deg 2$;

(ii) $f(F_f(X, Y)) = F_f(f(X), f(Y))$.

For each $a \in \mathcal{O}$, and $f, g \in \mathcal{F}_\pi$ we let $[a]_{f,g}(T)$ to be the series associated to the linear form $L = aX$ in Lemma 2.1.1. In particular, $[a]_{f,g}$ satisfies the following properties

(i) $[a]_{f,g}(T) \equiv aT \mod \deg 2$;

(ii) $f([a]_{f,g}(T)) = [a]_{f,g}(g(T))$.

The following theorem will justify the notation.

Theorem 2.2. *For series $f, g, h \in \mathcal{F}_\pi$ and elements $a, b \in \mathcal{O}$, the following identities hold:*

(i) $F_f(X, F_f(Y, Z)) = F_f(F_f(X, Y), Z)$;

(ii) $F_f(X, Y) = F_f(Y, X)$;

(iii) $F_f([a]_{f,g}(X), [a]_{f,g}(Y)) = [a]_{f,g}(F_g(X, Y))$;

(iv) $[a]_{f,g}([b]_{g,h}(T)) = [ab]_{f,h}(T)$;

(v) $[a + b]_{f,g}(T) = F_f([a]_{f,g}(T), [b]_{f,g}(T))$;

(vi) $[\pi]_f(T) = f(T)$, $\quad [1]_f(T) = T$.

In particular, for any $f \in \mathcal{F}_\pi$ we associate a formal group F_f called Lubin-Tate group for which f is an endomorphism. Analogously, the series $[a]_{f,g}$ coincides with the homomorphism $[a]_{F_f,F_g} \in Hom(F_f, F_g)$ previously defined in Theorem 2.1.

Proof. Straightforward application of unicity property in Lemma 2.1.1. □

We can observe that for every $f, g \in \mathcal{F}_\pi$, the formal groups F_f and F_g are canonically isomorphic over $\mathcal{O}$ through $[1]_{f,g}$. By this characterization, a formal group over $\mathcal{O}$ is in this isomorphism class if and only if it has an endomorphism reducing $\mod \pi$ to the Frobenius $T \mapsto T^q$, whose derivative at the origin is π.

Now observe that for each H algebraic extension of k with $\mathfrak{m}_H$ maximal ideal in the ring of integers of H, the set $\mathfrak{m}_H$ has an $\mathcal{O}$-module structure defined by

$$x + y = F_f(x, y)$$
$$\alpha \cdot x = [\alpha]_f(x)$$

for every $x, y \in \mathfrak{m}_H$ and every $\alpha \in \mathcal{O}$. Consider $\overline{k}$ the algebraic closure of k, then $\mathfrak{m}_{\overline{k}}$ has the structure of an $\mathcal{O}$-module denoted by $\mathfrak{m}_{\overline{k},f}$.

Definition 2.9. *For each $f \in \mathcal{F}_\pi$ and each integer $m \geq 1$ we let $\Lambda_{f,m}$ denote the $\mathcal{O}$-submodule of $\mathfrak{m}_{\overline{k},f}$ consisting of elements λ such that*

$$[\pi^m]_f \lambda = 0.$$

The following lemma will allow us to remove the dependence of $f \in \mathcal{F}_f$.

Lemma 2.1.2. *The field extension $k(\Lambda_{f,m})/k$ is a totally ramified Galois extension and it is independent of the choice $f \in \mathcal{F}_f$.*

Proof. Since the formal groups F_f, F_g are canonically isomorphism, we have $\lambda \in \Lambda_{f,m}$ if and only if $[1]_{g,f}(\lambda) \in \Lambda_{g,m}$. By completeness of field extension, we deduce $k(\Lambda_{f,m})$ and $k(\Lambda_{g,m})$ coincides. We can then consider f to be $\pi T + T^q$. Clearly, $k(\Lambda_{f,m})$ is the splitting field of $f^m(T)$ and then it is Galois. Furthermore, observe that $k(\Lambda_{f,m})$ contains the roots of the polynomial

$$[\pi^m]_f(T) = f^m(T) = X^{q^m} + \cdots + \pi^m T$$

and hence those of the polynomial

$$\Phi_m(T) = \frac{f^m(T)}{f^{m-1}(T)} = \frac{f(f^{m-1}(T))}{f^{m-1}(T)} = (f^{m-1}(T))^{q-1} + \pi$$

which is of degree $q^m - q^{m-1}$ and is irreducible over k by Eisenstein's criterion. We conclude $k(\Lambda_{f,m})/k$ is totally ramified. $\square$

We denote $k(\Lambda_{f,m})$ by $H_{\pi,m}/k$ and its Galois group by $G_{\pi,m}$. We let

$$\Lambda_f = \bigcup_{m \geq 1} \Lambda_{f,m}, \quad H_\pi = k(\Lambda_f), \quad G_\pi = \varprojlim_m G_{\pi,m}.$$

Theorem 2.3. *Let π be a prime element of $\mathcal{O}$ and let $f \in \mathcal{F}_\pi$. The following assertions hold*

(i) *the $\mathcal{O}$-module $\mathfrak{m}_{\overline{k},f}$ is divisible;*

(ii) *for each m, the $\mathcal{O}$-module $\Lambda_{f,m}$ is isomorphic to $\mathcal{O}/\mathfrak{p}^m$;*

(iii) *the $\mathcal{O}$-module Λ_f is isomorphic to $k/\mathcal{O}$;*

(iv) *for each $\tau \in G_\pi$ there exists a unique unit $u \in \mathcal{O}^\times$ such that for every $\lambda \in \Lambda_f$ we have*

$$\lambda^\tau = [u]_f(\lambda);$$

(v) *the map $\tau \mapsto u$ is an isomorphism of G_π onto $\mathcal{O}^\times$ under which the quotients $G_{\pi,m}$ of G_π correspond to the quotients $\mathcal{O}^\times/(1 + \pi^m \mathcal{O})$ of $\mathcal{O}^\times$;*

(vi) *for each* $m \geq 1$ *and for each generator* $\lambda_m \in \Lambda_m$*, the element* π *is the norm of* $-\lambda_m$ *for the extension* $H_{\pi,m}/k$*.*

Proof. In view of the isomorphism $[1]_{f,g}$, we may suppose $f(T) = \pi T + T^q$.

(i) Let $s \in \mathfrak{m}_{\overline{k}}$, we want to show that for each $\alpha \in \mathcal{O}$ there exists $r \in \mathfrak{m}_{\overline{k}}$ such that $[\alpha]r = s$. We can write $\alpha = u\pi^m$ with $u \in \mathcal{O}^\times$ and m positive integer. By 2.2, we have $[\alpha] = [u] \circ [\pi]^m$, and then we just need to prove the theorem for $\alpha = \pi$. Consider the polynomial $P_r(T) = T^q + \pi T - r$ and observe that since $r \in \mathfrak{m}_{\overline{k}}$ then all the roots of $P_r(T)$ are in $\mathfrak{m}_{\overline{k}}$. Let $t \in \mathfrak{m}_{\overline{k}}$ be a solution of $P_r(T) = 0$, we obtain

$$[\pi]t = f(t) = t^q + \pi t = r.$$

We conclude $\mathfrak{m}_{\overline{k}}$ is divisible.

(ii) The $\mathcal{O}$-module $\Lambda_{f,1}$, which consists of the roots of the equation $f(T) = T^q + \pi T = 0$, has q elements since $f(T)$ is coprime with $f'(T)$. Therefore, $\Lambda_{f,1}$ is a one-dimensional vector space over the residue field $\mathcal{O}/\mathfrak{p}$. For the general case observe that we have

$$[\pi^m]_f(T) = f^m(T) = X^{q^m} + \cdots + \pi^m T$$

and hence the polynomial

$$\Phi_m(T) = \frac{f^m(T)}{f^{m-1}(T)} = \frac{f(f^{m-1}(T))}{f^{m-1}(T)} = (f^{m-1}(T))^{q-1} + \pi$$

is of degree $q^m - q^{m-1}$ and is irreducible over k by Eisenstein's criterion. We deduce $\Lambda_{f,m}$ consists of q^m zeroes of $[\pi^m](T)$. Now if $\lambda_n \in \Lambda_{f,n} - \Lambda_{f,n-1}$, then

$$\begin{aligned} \mathcal{O} &\to \Lambda_n \\ a &\mapsto a \cdot \lambda_n \end{aligned}$$

is a homomorphism of $\mathcal{O}$-modules with kernel $\pi^n\mathcal{O}$. It induces a bijective homomorphism $\mathcal{O}/\pi^n\mathcal{O} \to \Lambda_n$ because both sides are of order q^n.

(iii) Follows from the previous point observing we have a compatible sequence

$$\cdots \xrightarrow{\pi} \Lambda_{f,2} \xrightarrow{[\pi]} \Lambda_{f,2} \xrightarrow{[\pi]} \Lambda_{f,1} \to 0.$$

(iv) An automorphism $\tau \in G_\pi$ induces an automorphism of the $\mathcal{O}$-module Λ_f. Indeed, if $\lambda \in \Lambda_{f,n}$ then $[\pi^n](\lambda) = 0$ and so $\tau([\pi^n](\lambda)) = 0$. Since $[\pi^n](T)$ has coefficients in $\mathcal{O}$ we deduce

$$([\pi^n](\lambda))^\tau = [\pi^n](\lambda^\tau) = 0$$

and then $\tau(\lambda) \in \Lambda_{f,n}$. By the previous point, we have $\Lambda_f \cong k/\mathcal{O}$, and for this module over a complete valuation ring $\mathcal{O}$, the only automorphism of Λ_f are those of the form $\lambda \mapsto [u]\lambda$ for $u \in \mathcal{O}^\times$.

(v) From the fact that Λ_f generates H_π, we deduce that the map $\tau \mapsto u$ is injective. More precisely, the unit u is congruent to $1 \mod \pi^m\mathcal{O}$ i.e. multiplication by u is identity on $(\mathcal{O}/\mathfrak{p}^m) \cong \Lambda_{f,m}$, if and only if τ is identity on $H_{\pi,m} = k(\Lambda_{f,m})$. Observe that $H_{\pi,m}$ contains the roots of the polynomial

$$[\pi^m]_f(T) = f^m(T) = X^{q^m} + \cdots + \pi^m T$$

and hence those of the polynomial

$$\Phi_m(T) = \frac{f^m(T)}{f^{m-1}(T)} = \frac{f(f^{m-1}(T))}{f^{m-1}(T)} = (f^{m-1}(T))^{q-1} + \pi$$

which is of degree $q^m - q^{m-1}$ and is irreducible over k by Eisenstein's criterion. Thus the order of $G_{\pi,m}$ is divided by $q^m - q^{m-1}$, that is the order of $\mathcal{O}^\times/(1+\mathfrak{p}^m)$ and the surjectivity follows. Passing to the inverse limit over m, we obtain $G_\pi \cong U$ because both groups are compact.

(vi) If λ_m is a root of the Eisenstein polynomial $\Phi_m(T)$ then we have $H_{\pi,m} = k(\lambda_m)$. In fact by the previous points every other element in $\Lambda_{f,n}$ is of the form $[a](\lambda_m)$ for a certain $a \in \mathcal{O}$. Since the Eisenstein polynomial is

$$\Phi_m(T) = (f^{m-1}(T))^{q-1} + \pi$$

we deduce $-\pi$ is the norm of λ_m for the extension $H_{\pi,m}/k$.

□

Let T be the maximal unramified extension of k, and let σ be the Frobenius automorphism of T over k. Since $H_{\pi,m}$ is the splitting field of an Eisenstein polynomial, we deduce H_π is totally ramified over k. In particular, H_π is linearly disjoint from T over k, and the Galois group $Gal(H_\pi T/k)$ is the product of $G_\pi = G(H_\pi/k)$ and $Gal(T/k)$.

Lemma 2.1.3. *For each prime element $\pi \in \mathcal{O}$, there exists an homomorphism*

$$r_\pi : k^\times \to Gal(H_\pi T/k)$$

such that for an arbitrary $a = u\pi^m \in k$ with $u \in \mathcal{O}^\times$ we have

(i) $r_\pi(a) = \sigma^m$ *on T;*

(ii) $\lambda^{r_\pi(a)} = [u^{-1}]_f(\lambda)$ *for $\lambda \in \Lambda_f$.*

Proof. We define for each unit $u \in \mathcal{O}^\times$, $r_\pi(u)$ to be the identity on T, and on H_π the reciprocal τ^{-1} of the element $\tau \in G_\pi$ corresponding to u. By the previous theorem, we have that this map is a homomorphism $r_\pi : \mathcal{O}^\times \to Gal(H_\pi T/k)$. We extend this map setting $r_\pi(\pi)$ to be the identity on H_π and the Frobenius automorphism σ on T. The properties (i) and (ii) are clearly verified. □

Theorem 2.4. *The field $H_\pi T$ and the homomorphism r_π are independent of π.*

Proof. See [LT65] Theorem 3. □

Theorem 2.5. *For the field $H_{\pi,n}/k$ of π^n-division points and for $a = u\pi^{v_k(a)} \in k^\times$, $u \in \mathcal{O}^\times$ we have*

$$r_{H_{\pi,n}/k}(a)(\lambda) = (a, H_{\pi,n}/k)(\lambda) = [u^{-1}]_f(\lambda)$$

for every $\lambda \in \Lambda_{f,n}$ and $r_{H_{\pi,n}/k}$ reciprocity map.

Proof. See [Neu99].V.5. Theorem 5.5. □

Corollary 2.5.1. *The field $H_{\pi,n}/k$ of π^n-division points is the class field relative to the group $(\pi) \times (1 + \pi^n \mathcal{O}) \subset K^\times$.*

Proof. For $a = u\pi^{v_k(a)}$ we have the following chain of equivalences

$$a \in N_{H_{\pi,n}/k}(H_{\pi,n}^\times) \iff (a, H_{\pi,n}/k) = 1 \iff [u^{-1}]_f(\lambda) = \lambda \text{ for all } \lambda \in \Lambda_{f,n}$$
$$\iff u^{-1} \in 1 + \pi^n\mathcal{O} \iff a \in (\pi) \times (1 + \pi^n\mathcal{O}).$$

□

Corollary 2.5.2. *The composite $H_\pi T$ is the maximal abelian extension of k.*

Proof. Let L/k be a finite abelian extension. Then we have $\pi^f \in N_{L/k}(L^\times)$ for a suitable f. Since $N_{L/k}(L^\times)$ is open in $k^\times$ and since the $(1 + \pi^n\mathcal{O})$ form a basis of neighborhoods of 1, we have $(\pi^f) \times (1 + \pi^n\mathcal{O}) \subseteq N_{L/k}(L^\times)$ for a suitable n. Hence L is contained in the class field of the group

$$(\pi^f) \times (1 + \pi^n\mathcal{O}) \subseteq N_{L/k}(L^\times) = ((\pi) \times (1 + \pi^n\mathcal{O})) \cap ((\pi^f) \times \mathcal{O}^\times).$$

The class field of $(\pi) \times (1 + \pi^n\mathcal{O})$ is $H_{\pi,n}$ by the previous corollary, while the class field of $(\pi^f) \times \mathcal{O}^\times$ is the unramified extension T_f of degree f. It follows that

$$L \subseteq H_{\pi,n}T_f \subseteq H_\pi T \subseteq k^{ab}.$$

□

2.2 Tower of fields

In this section, we will apply the results of Lubin-Tate theory to K quadratic imaginary extension. In particular, considering E the elliptic curve associated, we can consider its formal group defined over a completion of K. This structure will correspond exactly to the Lubin-Tate setting and it will allow us to study properly the tower field extension arising from K adjoint with $\mathfrak{p}$-torsion points.

2.2.1 Formal group of an elliptic curve

Definition 2.10. *Let E be an elliptic curve given by a Weierstrass equation with coefficients in R localization of $\mathcal{O}_K$ at $\mathfrak{p}$. The formal group associated with E, denoted $\widehat{E}$, is given by the formal power series associated with the group law on E of parameter $t = -2x/y = -2\wp(z)/\wp'(z) = \varepsilon(z)$.*

Observe that $\widehat{E}$ is defined over R but we consider it over its completion $\mathcal{O}_\mathfrak{p}$, in particular we have formal power series expansions

$$x = t^{-2}a(t), \quad y = -2t^{-3}a(t)$$

where $a(t)$ has coefficients in $\mathcal{O}_\mathfrak{p}$ and constant term equal to 1. Recall that from the theory of elliptic curves, there exists a translation invariant differential $\omega_E = dx/2y$. This in particular induces a formal invariant differential on $\widehat{E}$ (See [Sil09] IV). We can then define the formal logarithm λ of $\widehat{E}$ that induces an isomorphism between $\widehat{E}$ and $\widehat{\mathbb{G}}_a$

$$\lambda : \widehat{E} \xrightarrow{\sim} \widehat{\mathbb{G}}_a$$

according to Prop. 2.1.2. In particular from the relation $t = -2x/y = -2\wp(z)/\wp'(z) = \varepsilon(z)$, we can view z as being a parameter of the formal additive group $\mathbb{G}_a$ and then $\varepsilon(z)$ is the exponential map of $\widehat{E}$. Furthermore observe that every isogeny $[\alpha]$ of E for $\alpha \in \mathcal{O}_K$ induces a formal endomorphism

$$[\alpha] : \widehat{E} \to \widehat{E}$$

Lemma 2.2.1. *For every $\alpha \in \mathcal{O}_K$, we have*

$$[\alpha](t) \equiv \alpha t \mod \deg 2.$$

Proof. By definition of the invariant differential we have $\omega([\alpha](P)) = \alpha\omega(P)$, in particular we obtain

$$\frac{d(x([\alpha](t)))}{2y([\alpha](t))} = \alpha \frac{d(x(t))}{2y(t)}.$$

Using the definition of $x(t)$ and $y(t)$, the right-hand side is $(\alpha + O(t))dt$ and the left-hand side is $([\alpha]'(0) + O(t))dt$. This completes the proof. □

From the isomorphism λ we deduce

$$\lambda([\alpha](t)) = \alpha\lambda(t).$$

We write $\tilde{E}$ for the reduction of E modulo $\mathfrak{p}$. Recall the notation of the following subsets of $E(K_\mathfrak{p})$

$$E_0(K_\mathfrak{p}) = \{P \in E(K_\mathfrak{p}) : \tilde{P} \in \tilde{E}_{ns}(\mathcal{O}/\mathfrak{p})\};$$
$$E_1(K_\mathfrak{p}) = \{P \in E(K_\mathfrak{p}) : \tilde{P} = \tilde{O}\}.$$

We have the following standard results.

Prop. 2.2.1. *There is an exact sequence of abelian groups*

$$0 \to E_1(K_\mathfrak{p}) \to E_0(K_\mathfrak{p}) \to \tilde{E}_{ns}(\mathcal{O}/\mathfrak{p}) \to 0$$

where the right-hand map is the reduction modulo π.

Proof. See [Sil09] VII.2.1. □

Prop. 2.2.2. *Let $E/K_\mathfrak{p}$ be given by a minimal Weierstrass equation, let $\widehat{E}/R$ be the formal group associated to E. Then there is an isomorphism*

$$\hat{E}(\mathfrak{p}) \xrightarrow{\sim} E_1(K_\mathfrak{p})$$

Proof. See [Sil09] VII.2.2. □

Theorem 2.6. *$\widehat{E}$ is a Lubin-Tate group over $K_\mathfrak{p}$. It is of height 1 if $\mathfrak{p}$ splits in $K/\mathbb{Q}$ and of height 2 if $\mathfrak{p}$ is inert or ramified.*

Proof. Let φ the Frobenius automorphism. The isogeny $\psi(\mathfrak{p}) : E \to E^\varphi$ associated to $\mathfrak{p}$, induces a homomorphism of formal groups

$$\widehat{\psi(\mathfrak{p})} : \hat{E} \to \hat{E}$$

which is of the form

$$\widehat{\psi(\mathfrak{p})}(T) = \pi T + \cdots \in \mathcal{O}_\mathfrak{p}$$
$$\widehat{\psi(\mathfrak{p})}(T) \equiv T^q \mod \mathfrak{p}\mathcal{O}_\mathfrak{p}$$

with $q = N\mathfrak{p}$. Indeed, by Corollary B.3.1.(iii) we have that $\psi(\mathfrak{p})$ reduces modulo $\mathfrak{p}$ to the Frobenius automorphism φ_q of $\tilde{E}$ which corresponds to the formal endomorphism T^q. Since the formal group $\hat{E}$ has an endomorphism of the form of definition 2.7, then by Theorem 2.2 we conclude $\widehat{E}$ is a Lubin-Tate group. Observe that by Theorem 2.1 we have $\widehat{\psi(\mathfrak{p})}$ coincide with the endomorphism $[\pi]$. □

The following lemma uses the theory just encountered to deduce properties about the order of coordinate functions of E. It will be very useful later in the study of the theory of Theta functions.

Lemma 2.2.2. *Let $\mathfrak{b}, \mathfrak{c}$ be non trivial coprime ideals of $\mathcal{O}_K$ and $P \in E_\mathfrak{b}$, $Q \in E_\mathfrak{c}$ primitive torsion points in $E(\overline{K})$ Fix an extension of the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$ to $\overline{K}$ normalized to $v_\mathfrak{p}(\mathfrak{p}) = 1$.*

(i) *If $n > 0$ and $\mathfrak{b} = \mathfrak{p}^n$ then $v_\mathfrak{p}(x(P)) = -2/(q^n - q^{n-1})$.*

(ii) *If $\mathfrak{b}$ is not a power of $\mathfrak{p}$ then $v_\mathfrak{p}(x(P)) \geq 0$.*

(iii) *If $\mathfrak{p} \nmid \mathfrak{b}\mathfrak{c}$ then $v_\mathfrak{p}(x(P) - x(Q)) = 0$.*

Proof. (i) Suppose $\mathfrak{b} = \mathfrak{p}^n$ with $n \geq 1$. Let $\hat{E}$ be the formal group over $\mathcal{O}_{K_\mathfrak{p}}$. Let $\pi = \psi(\mathfrak{p})$, by the previous theorem, we have

$$\begin{cases} [\pi](T) \equiv \pi T \mod \deg 2 \\ [\pi](T) \equiv T^q \mod \mathfrak{p}\mathcal{O}_\mathfrak{p}. \end{cases}$$

Define $f(T) \in \mathcal{O}_{K_\mathfrak{p}}$ to be the power series

$$f(T) = \frac{[\pi^n](T)}{[\pi^{n-1}](T)} = \frac{[\pi][\pi^{n-1}](T)}{[\pi^{n-1}](T)}$$

and then we have

$$\begin{cases} f(T) \equiv \pi \mod T \\ f(T) \equiv T^{q^n - q^{n-1}} \end{cases}.$$

Thus, by Weierstrass preparation theorem 5.1,

$$f(T) = e(T)u(T)$$

where $e(T)$ is an Eisenstein polynomial of degree $q^n - q^{n-1}$ and $u(x) \in \mathcal{O}[[T]]^\times$. By Prop 2.2.2 we have $E_{\mathfrak{p}^n} \subset E_1(\overline{K}_\mathfrak{p})$, so we get for $P \in E_{\mathfrak{p}^n}$ that $z = -x(P)/y(P)$ is a root of $f(T)$ and hence of $e(T)$,

$$v_\mathfrak{p}(x(P)/y(P)) = (q^n - q^{n-1})^{-1}.$$

Since for every $(x(P), y(P)) \in E_1(\overline{K}_\mathfrak{p})$ we have $3v_\mathfrak{p}(x(P)) = 2v_\mathfrak{p}(y(P))$, then we conclude

$$v_\mathfrak{p}(x(P)) = -2/(q^n - q^{n-1}).$$

(ii) If $\mathfrak{b}$ is not a power of $\mathfrak{p}$ then by Prop. 2.2.2 we have $P \notin E_1(\overline{K}_\mathfrak{p})$. Hence we conclude $v_\mathfrak{p}(x(P)) \geq 0$.

(iii) Let $\tilde{P}, \tilde{Q}$ the reductions modulo $\mathfrak{p}$ of P and Q. Then we have

$$v_\mathfrak{p}(x(P) - x(Q)) > 0 \iff x(\tilde{P}) = x(\tilde{Q}) \iff \tilde{P} = \pm\tilde{Q} \iff P \pm Q \in E_1(\overline{K}_\mathfrak{p}).$$

Since $\mathfrak{b}, \mathfrak{c}$ are coprime, then the order of $P \pm Q$ is not a power of $\mathfrak{p}$. So again, by Prop. 2.2.2 we obtain $P \pm Q \notin E_1(\overline{K}_\mathfrak{p})$.

□

We now focus on the case in which $\mathfrak{p}$ is a split prime and the Lubin Tate group $\hat{E}$ is of height 1. By Lemma 2.1.1, there is a unique formal. group $\mathcal{E}$ defined over $\mathcal{O}_\mathfrak{p}$ such that the endomorphism $[\pi]$ of $\mathcal{E}$ is given by the power series $[\pi](w) = \pi w + w^p$. Moreover, by Theorem 2.2, the formal group $\mathcal{E}$ is canonically isomorphic to $\hat{E}$ over $\mathcal{O}_\mathfrak{p}$. Denote $\mathcal{E}_{\pi^{n+1}}$ the kernel of the endomorphism $[\pi^{n+1}]$ of $\mathcal{E}$. Then we can rewrite Lemma 2.1.2 and Theorem 2.3 as follows.

Theorem 2.7. *For each $n \geq 0$, $K_\mathfrak{p}(E_{\pi^{n+1}}) = K_\mathfrak{p}(\mathcal{E}_{\pi^{n+1}})$ is a totally ramified extension of degree $p^n(p-1)$. If u_n is a generator of $E_{\pi^{n+1}}$, then its norm is equal to $-\pi$. In particular, u_n is a local parameter for $K_\mathfrak{p}(E_{\pi^{n+1}})$. Furthermore, we have*

i) *the* $\mathcal{O}_{\mathfrak{p}}$*-module* E_{π^n} *is isomorphic to* $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n$*,*

ii) *the* $\mathcal{O}_{\mathfrak{p}}$*-module* E_{π^∞} *is isomorphic to* $K_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}$*,*

iii) $Gal(K_{\mathfrak{p}}(E_{\pi^n})/K_{\mathfrak{p}})$ *is isomorphic to* $\mathcal{O}_{\mathfrak{p}}^{\times}/(1+\pi^n\mathcal{O}_{\mathfrak{p}})$ *and* $Gal(K_{\mathfrak{p}}(E_{\pi^\infty})/K_{\mathfrak{p}})$ *is isomorphic to* $\mathcal{O}_{\mathfrak{p}}^{\times}$*.*

2.2.2 Ray class fields and extensions with torsion points

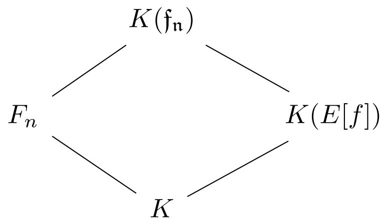
Lemma 2.2.3. *Let* $\mathfrak{f} = (f)$ *be the conductor of the Hecke character* ψ *of* E*, and let* E_f *be the group of* f*-division points on* E*. Then the field* $K(E_f)$ *coincides with the ray class field of* K *modulo* $\mathfrak{f}$*.*

Proof. By Theorem B.2 we have that $K(\mathfrak{f}) = K(h(E_f)) \subseteq K(E_f)$. Let now $\mathbb{A}_K^{\times}$ be the idele group of K, and let $U(\mathfrak{f})$ be the subgroup of $\mathbb{A}_K^{\times}$ corresponding to the ray class field modulo $\mathfrak{f}$. Let $\xi(\tau) = (\wp(\tau), \wp'(\tau))$ be an arbitrary f-division point on E. Let $x \in U(\mathfrak{f})$ with $x_\infty = 1$. We must show that the Artin symbol $(x, K^{ab}/K)$ of x fixes $\xi(\tau)$. By Theorem B.3.1.(ii) we have

$$\xi(\tau)^{(x,K^{ab}/K)} = \xi(\psi(x)\tau).$$

But, as $\mathfrak{f}$ is the conductor of ψ and $x \in U(\mathfrak{f})$ with $x_\infty = 1$, we have $\psi(x) = 1$. Hence, $(x, K^{ab}/K)$ fixes $\xi(\tau)$ for any $x \in U(\mathfrak{f})$ that completes the proof of the lemma. □

Lemma 2.2.4. *For each integer* $n \geq 0$*, the conductor of* $F_n = K(E_{\overline{\pi}^{n+1}})$ *over* K *is* $\mathfrak{f}_n = \mathfrak{f}\overline{\mathfrak{p}}^{n+1}$*. Moreover, the ray class field* $K(\mathfrak{f}_n)$ *modulo* $\mathfrak{f}_n$ *is the compositum of* F_n *and* $H = K(E_f)$*, and* $F_n \cap H = K$*.*



Proof. See [CW77] Lemma 4. □

We introduce now two important field extensions that will be largely used in the rest of the mémoire.

Definition 2.11. *For each pair of integers* $m, n \geq 0$ *we define*

$$F_m = K(E_{\overline{\pi}^{m+1}}),$$
$$K_{n,m} = F_m(E_{\pi^n}).$$

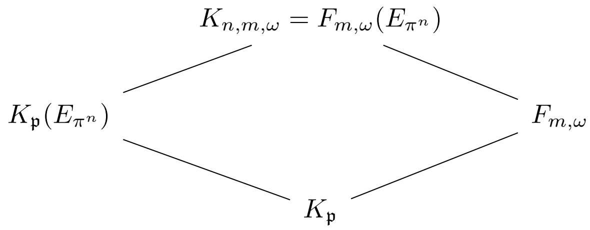
Prop. 2.2.3. *With the previous notation we have the following properties.*

(i) *The extension* F_m/K *is unramified at* $\mathfrak{p}$*.*

(ii) *The extension* $K_{n,m}/F_m$ *is totally ramified at the primes above* $\mathfrak{p}$*.*

Proof. (i) By Lemma 2.2.4, we have that F_m is contained in the ray class field of K modulo $\mathfrak{f}\overline{\mathfrak{p}}^{m+1}$. In particular, F_m is unramified at $\mathfrak{p}$.

(ii) First of all notice that we can reduce to study the extension locally. Let ω a prime of F_m lying above $\mathfrak{p}$.



By Theorem 2.3 applied to the elliptic formal group, we have that the extension $K_{\mathfrak{p}}(E_{\pi^n})/K_{\mathfrak{p}}$ is the splitting field of an Eisenstein polynomial of the form

$$X^{p^{n-1}(p-1)} + \cdots + \pi.$$

Since F_m/K is unramified we deduce that the polynomial is Eisenstein over F_m and then the extension $F_{m,\omega}(E_{\pi^n})/F_{m,\omega}$ is totally ramified generated by the same polynomial.

□

The following lemma gives us the decomposition of $\mathfrak{p}$ in the previous extensions.

Lemma 2.2.5. *Let r_m be the number of primes of F_m lying above $\mathfrak{p}$. Then r_m is given by the index of the subgroup generated by π in $(\mathcal{O}_K/\overline{\pi}^{m+1})^\times$. In particular, there exists an integer M such that*

$$r_m = \begin{cases} r_0 p^m & \text{for } m < M \\ r_0 p^M & \text{for } m \geq M. \end{cases}$$

Proof. First of all, observe that by Corollary B.3.2.(ii) we have

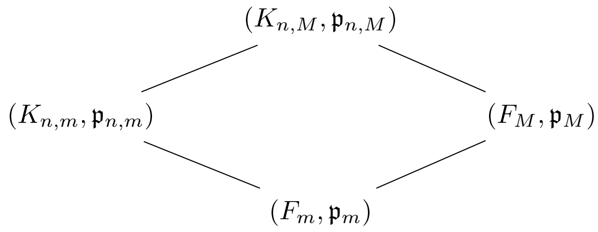
$$Gal(F_m/K) \xrightarrow{\sim} (\mathcal{O}_K/\overline{\pi}^{m+1})^\times.$$

By Corollary B.3.1.(ii), we have that $(\mathfrak{p}, F_m/K)$ acts on $E_{\overline{\mathfrak{p}}^{m+1}}$ by multiplication by π. In particular, we have that in the previous isomorphism

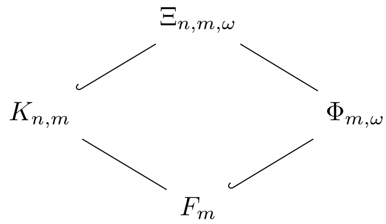
$$(\mathfrak{p}, F_m/K) \mapsto \pi.$$

By the previous proposition, $\mathfrak{p}$ is unramified in F_m and then the number of primes lying above $\mathfrak{p}$ is given by the index of $(\mathfrak{p}, F_m/K)$ in $Gal(F_m/K)$ and then of π in $(\mathcal{O}_K/\overline{\pi}^{m+1})^\times$. Since $\varprojlim(\mathcal{O}_K/\overline{\pi}^n) \cong \mathbb{Z}_p^\times \cong \mu_p \times (1 + p\,\mathbb{Z}_p)$ with μ_p p-roots of unity, we have that π has index $r_0 p^M$ for a certain $M \geq 0$. □

We then choose and fix $\mathfrak{p}_M$ prime of F_M lying above $\mathfrak{p}$ and let $\mathfrak{p}_m$ denote the unique prime of F_m lying above or below $\mathfrak{p}_M$. In this way, we have fixed our choice of extensions in the tower field. Since $K_{n,m}$ over F_m is totally ramified by the previous Lemma, we write $\mathfrak{p}_{n,m}$ for the unique prime of $K_{n,m}$ lying above $\mathfrak{p}_m$.



If ω is any prime of F_m lying above $\mathfrak{p}$, we let $\Xi_{n,m,\omega}$ be the completion of $K_{n,m}$ at the unique prime above ω, and we let $\Phi_{m,\omega}$ denote the completion of F_m at ω. We always view our global fields equipped with embeddings into their completions. We write $\mathcal{R}_{m,\omega}$ for the ring of integers of $\Phi_{m,\omega}$ and we use ω for the maximal ideal of $\mathcal{R}_{m,\omega}$.



For simplicity, we shall omit the subscript for the prime when referring to the completions at or above $\mathfrak{p}_m$ as previously fixed. Let $K_{\mathfrak{p}}$ be the completion of K at $\mathfrak{p}$, and we shall identify its ring of integers $\mathcal{O}_{\mathfrak{p}}$ with $\mathbb{Z}_{\mathfrak{p}}$. We also define the following fields

$$K_\infty = \bigcup_{n,m\geq 0} K_{n,m}, \quad F_\infty = \bigcup_{m\geq 0} F_m, \quad \Phi_\infty = \bigcup_{m\geq 0} \Phi_m.$$

Let φ denote the Artin symbol $(\mathfrak{p}, F_\infty/K)$ for the extension F_∞ over K. By the previous Lemma, we have F_∞ is unramified over K, this implies that φ induces the Frobenius automorphism for the extension $\Phi_\infty/K_{\mathfrak{p}}$.

We then define the rings

$$\Xi_{n,m} = \prod_{\omega} \Xi_{n,m,\omega}, \quad \Phi_m = \prod_{\omega} \Phi_{m,\omega}, \quad \mathcal{R}_m = \prod_{\omega} \mathcal{R}_{m,\omega}$$

where the product is taken over the set of primes ω of F_m lying above $\mathfrak{p}$. The Galois group $G_\infty = Gal(K_\infty/K)$ acts naturally on these rings as follows. Let $(\alpha_{\omega,k})_k$ be a Cauchy sequence of elements of $K_{n,m}$ (or F_m) which converge to α_ω in $\Xi_{n,m,\omega}$ (or $\Phi_{m,\omega}$). Then the ω^σ component of $(\alpha_\omega)^\sigma$ is the limit of the Cauchy sequence $(\alpha^\sigma_{k,\omega})$ in Ξ_{n,m,ω^σ} (or Φ_{m,ω^σ}). We embed $K_{n,m}$ and F_m in these rings via the diagonal map. The usual norm and trace maps on $\Xi_{n,m}$, Φ_m, $K_{n,m}$, and F_m as well as the Galois action, all commute with these embeddings.

We denote by $U'_{n,m,\omega}$ the units of $\Xi_{n,m,\omega}$ and by $U_{n,m,\omega}$ the subgroup consisting of those units which are congruent to 1 modulo the maximal ideal. We then define

$$U'_{n,m} = \prod_{\omega} U'_{n,m,\omega}, \quad U_{n,m} = \prod_{\omega} U_{n,m,\omega}$$

where the product is taken over the set of primes ω of F_m lying above $\mathfrak{p}$. Furthermore, we introduce the multiplicative groups

$$U'_\infty = \varprojlim U'_{n,m}, \quad U_\infty = \varprojlim U_{n,m}$$

where the projective limit is taken with respect to the norm maps on the $\Xi_{n,m}$. We endow U_∞ with its natural structure as a $\mathbb{Z}_p[G_\infty]$-module.

2.2.3 Action of G_∞ and structure of $\mathbb{Z}_p[\Lambda]$-module

We denote G_∞ the Galois group of K_∞/K. From the theory of Lubin-Tate groups we deduce the following properties of the action of G_∞.

Lemma 2.2.6. *The action of G_∞ on E_{π^∞} and $E_{\overline{\pi}^\infty}$ induces two characters*

$$k_1 : G_\infty \to \mathbb{Z}_p^\times, \quad k_2 : G_\infty \to \mathbb{Z}_p^\times$$

with the property that for $\sigma \in G_\infty$ and $\alpha \in \mathcal{O}_K$ such that

$$u^\sigma = \alpha u$$

for all $u \in E_{\overline{\pi}^{m+1}}$ then $k_2(\sigma)$ is given by

$$k_2(\sigma) \equiv \alpha \mod \overline{\mathfrak{p}}^{m+1}, \quad k_2(\sigma) \equiv \overline{\alpha} \mod \mathfrak{p}^{m+1}$$

where we have used the identification of $\mathbb{Z}_p$ with $\mathcal{O}_\mathfrak{p}$.

Proof. First of all observe that the extensions F_∞/K and $K(E_{\pi^\infty})/K$ are totally ramified respectively at $\overline{\mathfrak{p}}$ and $\mathfrak{p}$. Then the global Galois group coincides with the local one. By Theorem 2.3 applied to the elliptic curve case, we have

$$Gal(F_\infty/K) \xrightarrow{\sim} (\varprojlim_n \mathcal{O}/\overline{\mathfrak{p}}^n)^\times \cong \mathbb{Z}_p^\times,$$
$$Gal(K(E_{\pi^\infty})/K) \xrightarrow{\sim} (\varprojlim_n \mathcal{O}/\mathfrak{p}^n)^\times \cong \mathbb{Z}_p^\times$$

that define the maps k_1, k_2. Observe that the theorem depends on the fact that E_{π^n} is a free $\mathcal{O}/\mathfrak{p}^n$-module of rank 1. If for all $u \in E_{\overline{\pi}^{m+1}}$ we have $u^\sigma = \alpha u$ for $\sigma \in G_\infty$ and $\alpha \in \mathcal{O}_K$ we have that σ correspond to an element in $(\mathcal{O}/\overline{\mathfrak{p}}^{m+1})^\times$ that could be represented by an integer modulo p^{m+1} through the isomorphism $\mathcal{O}/\overline{\mathfrak{p}}^{m+1} \cong \mathbb{Z}/p^{m+1}\mathbb{Z}$. □

Theorem 2.8. *The Galois group of G_∞ is of the form*

$$G_\infty = \Gamma \times \Delta$$

where $\Gamma = \mathbb{Z}_p \times \mathbb{Z}_p \cong Gal(K_\infty, K_{0,0})$ and $\Delta = (\mathbb{Z}/(p-1)\mathbb{Z}) \times (\mathbb{Z}/(p-1)\mathbb{Z}) \cong Gal(K_{0,0}/K)$.

Proof. By the previous Lemma, we have that $Gal(K[\overline{\pi}^\infty]/K)$ and $Gal(K[\pi^\infty]/K)$ are isomorphic to $\mathbb{Z}_p^\times$. Furthermore observe that from Corollary B.3.2.(ii) we have

$$Gal(K(E[\pi^{n+1}\overline{\pi}^{m+1}]/K) = Gal(K_{n,m}/K) \xrightarrow{\sim} (\mathcal{O}_K/\mathfrak{p}^{n+1}\overline{\mathfrak{p}}^{m+1})^\times \cong$$
$$\cong (\mathcal{O}_K/\mathfrak{p}^{n+1})^\times \times (\mathcal{O}_K/\overline{\mathfrak{p}}^{m+1})^\times$$

Since K_∞ is generated by $K(E_{\overline{\pi}^\infty}), K(E_{\pi^\infty})$ we deduce

$$(k_1, k_2) : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times .$$

Furthermore, we have

$$Gal(K_{0,0}/K) \xrightarrow{\sim} (\mathcal{O}_K/\mathfrak{p}\overline{\mathfrak{p}})^\times \cong (\mathbb{Z}/p\,\mathbb{Z})^\times \times (\mathbb{Z}/p\,\mathbb{Z})^\times = \Delta.$$

From the isomorphism $\mathbb{Z}_p^\times \cong (\mathbb{Z}/(p-1)\,\mathbb{Z}) \times \mathbb{Z}_p$ we conclude

$$\Gamma = \mathbb{Z}_p \times \mathbb{Z}_p \cong Gal(K_\infty, K_{0,0}).$$

□

We denote by χ_1, χ_2 the restriction of k_1, k_2 to Δ. Together they generate $Hom(\Delta, \mathbb{Z}_p^\times)$.

Let $\Lambda = \mathbb{Z}_p[[T_1, T_2]]$ be the ring of formal power series in two indeterminates. Let u a topological generator of $(1 + p\,\mathbb{Z}_p)$ and let γ_1, γ_2 two elements of Γ for which

$$k_1(\gamma_1) = u, \quad k_2(\gamma_1) = 1,$$
$$k_1(\gamma_2) = 1, \quad k_2(\gamma_2) = u.$$

In particular, any $\mathbb{Z}_p$-module B on which Γ acts continuously can be endowed with a unique Λ-module structure such that

$$\gamma_1 x = (1 + T_1)x,$$
$$\gamma_2 x = (1 + T_2)x$$

for all $x \in B$.

CHAPTER 3

Elliptic units

This chapter is the essential core of the whole mémoire. We define the elliptic units and we study the relation with special values of L-functions. The elliptic units will be defined as certain rational functions of x-coordinates of torsion points on a CM elliptic curve. In particular, they are units in abelian extension of quadratic imaginary fields, they play a role analogous to that of circular units in abelian number fields.

The bridge between special values of Hecke L series associated with K and elliptic units is provided by special values of Eisenstein series. Following De Shalit's [DS87] approach we will call them Eisenstein numbers since their role parallels the role Bernoulli numbers play in the cyclotomic theory.

3.1 Theta functions

We start studying the rational function $\Theta_{E,\mathfrak{a}}$ on the elliptic curve. We will explore its properties from both the algebraic point of view and from the complex analytical one.

3.1.1 Theta function over K

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ and assume K has class number 1. Consider E an elliptic curve defined over K with complex multiplication by $\mathcal{O}_K$. Let S be the set of finite primes q with bad reduction together with 2,3. Fix a Weierstrass model for E

$$y^2 = 4x^3 - g_2x - g_3 \tag{3.1}$$

such that the discriminant of (3.1) is divisible only by primes of K lying above primes in S (See [Sil94].VIII.8). In particular, let $\wp(z)$ be the Weierstrass function associated with (3.1) and L the period lattice of $\wp(z)$. Then we have a group isomorphism

$$\begin{aligned} \xi : \mathbb{C}/L &\to E(\mathbb{C}) \\ z &\mapsto (\wp(z), \wp'(z)). \end{aligned}$$

As usual, we identify $End_K(E)$ with $\mathcal{O}$ in such a way that the endomorphism corresponding to $\alpha \in \mathcal{O}$ is $\xi(z) \mapsto \xi(\alpha z)$. Furthermore, there exists $\Omega_\infty \in \mathbb{C}$ such that $L = \Omega_\infty \mathcal{O}$. Let $\mathfrak{a} \subseteq \mathcal{O}_K$ be an integral ideal of K prime to the ideals in S. We denote $N\mathfrak{a}$ the absolute norm of $\mathfrak{a}$ and $\mathfrak{a}^{-1}L$ denotes the lattice $\Omega_\infty \mathfrak{a}^{-1}$.

Definition 3.1. *Define $\Theta_{E,\mathfrak{a}}$ a rational function on E with coordinate functions x, y by*

$$\Theta_{E,\mathfrak{a}} := \alpha^{-12}\Delta(E)^{N\mathfrak{a}-1} \prod_{P \in E_\mathfrak{a} - O} (x - x(P))^{-6}. \tag{3.2}$$

Lemma 3.1.1. *Let $\mathfrak{a}$ be an integral ideal of K*

(i) $\Theta_{E,\mathfrak{a}}$ *is independent of the choice of the Weierstrass model.*

(ii) *if $\varphi : E' \to E$ is an isomorphism of elliptic curves then $\Theta_{E',\mathfrak{a}} = \Theta_{E,\mathfrak{a}} \circ \varphi$.*

(iii) $\Theta_{E,\mathfrak{a}}$ *is a rational function on E defined over K with divisor*

$$12N\mathfrak{a}[O] - 12 \sum_{P \in E_\mathfrak{a}} [P]. \tag{3.3}$$

Proof. (i) Any other Weierstrass model has coordinate functions x', y' given by

$$\begin{cases} x' &= u^2 x + r \\ y' &= u^3 y + sx + t \end{cases}$$

with $u \in \mathbb{C}^\times$, and then $a_i' = u^i a_i$ and

$$\Delta(E') = u^{12}\Delta(E).$$

By Prop. B.0.1 we have $\#E_\mathfrak{a} = N\mathfrak{a}$ and then we obtain

$$\Theta_{E',\mathfrak{a}} = \alpha^{-12}\Delta(E')^{N\mathfrak{a}-1} \prod_{P \in E_\mathfrak{a} - O} (x' - x'(P))^{-6} =$$
$$= \alpha^{-12}\Delta(E)^{N\mathfrak{a}-1} u^{12N\mathfrak{a}-1} \prod_{P \in E_\mathfrak{a} - O} u^{-12}(x - x(P))^{-6} = \Theta_{E,\mathfrak{a}}.$$

(ii) In the previous point we showed $\Theta_{E,\mathfrak{a}}$ is independent of the Weierstrass model. Fix a Weierstrass model for E with coordinate function x, y and consider the Weierstrass model induced by φ with coordinate functions $x' = x \circ \varphi, y' = y \circ \varphi$. In particular we have $\Delta(E) = \Delta(E')$ Then applying φ we get

$$\Theta_{E,\mathfrak{a}} \circ \varphi = \alpha^{-12}\Delta(E)^{N\mathfrak{a}-1} \prod_{P \in E_\mathfrak{a} - O} (x \circ \varphi - x(P))^{-6} = \Theta_{E',\mathfrak{a}}.$$

(iii) Observe that $\alpha \in K$, $\Delta(E) \in K$ and $Gal(\overline{K}/K)$ permutes the set $\{x(P) : P \in E_\mathfrak{a} - O\}$ so $Gal(\overline{K}/K)$ fixes $\Theta_{E,\mathfrak{a}}$. The coordinate function x is an even rational function with a double pole at O and no other poles. Thus for every point P, the divisor of $x - x(P)$ is $[P] + [-P] - 2[O]$. Since $\#E_\mathfrak{a} = N\mathfrak{a}$ we conclude

$$div(\Theta_{E,\mathfrak{a}}) = \sum_{P \in E[\mathfrak{a}] - O} (6[P] + 6[-P] - 12[O]) = 12N\mathfrak{a}[O] - 12 \sum_{P \in E_\mathfrak{a}} [P].$$

∎

The following theorem is the fundamental result of this section and it will be the key in the construction of the elliptic units.

Theorem 3.1. *Let $\mathfrak{b}$ a nontrivial ideal of $\mathcal{O}_K$ relatively prime to $\mathfrak{a}$, let $Q \in E_{\mathfrak{b}}$ be a primitive $\mathfrak{b}$-division point.*

(i) $\Theta_{E,\mathfrak{a}}(Q) \in K(\mathfrak{b})$.

(ii) *If $\mathfrak{c}$ is an ideal of $\mathcal{O}_K$ prime to $\mathfrak{b}$, c a generator of $\mathfrak{c}$, and $\sigma_{\mathfrak{c}} = (\mathfrak{c}, K(\mathfrak{b})/K)$ its Artin symbol. Then we have*

$$\Theta_{E,\mathfrak{a}}(Q)^{\sigma_{\mathfrak{c}}} = \Theta_{E,\mathfrak{a}}(cQ)$$

(iii) *If $\mathfrak{b}$ is not a prime power then $\Theta_{E,\mathfrak{a}}(Q)$ is a global unit of $K(\mathfrak{b})$. If $\mathfrak{b}$ is a power of $\mathfrak{p}$ then $\Theta_{E,\mathfrak{a}}(Q)$ is a unit at primes not dividing $\mathfrak{p}$.*

Proof. (i) By the previous lemma, we have $\Theta_{E,\mathfrak{a}}$ belongs to the function field $K(E)$. Let ψ be the Hecke character associated to E by Theorem B.3. Then consider $x \in U(\mathfrak{b}) \subset \mathbb{A}_K^\times$ an element in the Ray class group modulo $\mathfrak{b}$ with $x_\infty = 1$. In particular we have $x \equiv 1 \mod {}^\times\mathfrak{b}$ and let $\sigma_x = (x, K^{ab}/K)$ its Artin symbol. By Corollary B.3.1 we have

$$Q^{\sigma_x} = \psi(x)Q.$$

where $\psi(x) \in \mathcal{O}_K^\times$ induces an automorphism of E through the isogeny associated. Therefore we obtain the chain of equalities

$$\Theta_{E,\mathfrak{a}}(Q)^{\sigma_x} = \Theta_{E,\mathfrak{a}}(Q^{\sigma_x}) = \Theta_{E,\mathfrak{a}}(\psi(x)Q) = \Theta_{E,\mathfrak{a}}(Q)$$

where the last equality follows by Lemma 3.1.1.(ii).

(ii) Let now $x \in \mathbb{A}_K^\times$ be an idele with $x\mathcal{O} = \mathfrak{c}$ and $x_{\mathfrak{p}} = 1$ for $\mathfrak{p}$ dividing $\mathfrak{b}$. Then by Corollary B.3.1 we have $\psi(x) \in c\mathcal{O}^\times$ and $\psi(x)Q = Q^{\sigma_{\mathfrak{c}}}$. So again using Lemma 3.1.1.(ii),

$$\Theta_{E,\mathfrak{a}}(Q)^{\sigma_{\mathfrak{c}}} = \Theta_{E,\mathfrak{a}}(\psi(x)Q) = \Theta_{E,\mathfrak{a}}(cQ).$$

(iii) Let $\mathfrak{p}$ be a prime of K such that $\mathfrak{b}$ is not a power of $\mathfrak{p}$. Up to consider a different Weierstrass model, we can assume E has good reduction at $\mathfrak{p}$, so that $\Delta(E)$ is prime to $\mathfrak{p}$. Let $n = v_{\mathfrak{p}}(\mathfrak{a})$. Then

$$\begin{aligned} v_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q)) &= -12n - 6 \sum_{P \in E_{\mathfrak{a}} - O} v_{\mathfrak{p}}(x(Q) - x(P)) = \\ &= -12n - 6 \sum_{P \in E_{\mathfrak{p}^n} - O} v_{\mathfrak{p}}(x(Q) - x(P)) - \\ &\quad - 6 \sum_{P \in E_{\mathfrak{a}} - E_{\mathfrak{p}^n}} v_{\mathfrak{p}}(x(Q) - x(P)). \end{aligned}$$

By Lemma 2.2.2, since $\mathfrak{b}$ is not a power of $\mathfrak{p}$ we get

$$v_{\mathfrak{p}}(x(Q) - x(P)) = \\ = \begin{cases} -2/(N\mathfrak{p}^m - N\mathfrak{p}^{m-1}) & \text{if } P \text{ has order exactly } \mathfrak{p}^m, m > 0 \\ 0 & \text{if the order of } P \text{ is not a power of } \mathfrak{p}. \end{cases}$$

From this we get

$$v_{\mathfrak{p}}(\Theta_{E,\mathfrak{a}}(Q)) = -12n - 6\sum_{i=1}^{n}\sum_{P\in E_{\mathfrak{p}^i}-E_{\mathfrak{p}^{i-1}}} v_{\mathfrak{p}}(x(Q)-x(P)) = 0.$$

Observe that if $\mathfrak{b}$ is not a prime power then $\Theta_{E,\mathfrak{a}}(Q)$ is unit for every prime. Otherwise, if $\mathfrak{b} = \mathfrak{p}^n$ then $\Theta_{E,\mathfrak{a}}(Q)$ is a unit outside of $\mathfrak{p}$.

□

Theorem 3.2 (Distribution relation). *Let $\mathfrak{b}$ an integral ideal of K, relatively prime to $\mathfrak{a}$, let $b \in \mathcal{O}_K$ any generator of $\mathfrak{b}$. Then we have*

$$\prod_{R\in E_{\mathfrak{b}}} \Theta_{E,\mathfrak{a}}(P+R) = \Theta_{E,\mathfrak{a}}(\beta P)$$

where the product is taken over the $\mathfrak{b}$-torsion points.

Proof. First of all, observe that both sides of the equation are rational functions on E. By Lemma 3.1.1, we can compute the divisors

$$div(\Theta_{E,\mathfrak{a}}(P+R)) = 12N\mathfrak{a}[-R] - 12\sum_{Q\in E_{\mathfrak{a}}}[Q-R]$$

$$div\left(\prod_{R\in E_{\mathfrak{b}}} \Theta_{E,\mathfrak{a}}(P+R)\right) = \sum_{R\in E_{\mathfrak{b}}} 12N\mathfrak{a}[R] - 12\sum_{R\in E\mathfrak{b}}\sum_{Q\in E\mathfrak{a}}[Q+R]$$

for $R \in E_{\mathfrak{b}}$. Since $\mathfrak{a}, \mathfrak{b}$ are coprime we deduce $[Q+R]$ runs over all $\mathfrak{ab}$-torsion points for $Q \in E_{\mathfrak{a}}$ and $R \in E_{\mathfrak{b}}$. We can then rewrite

$$div\left(\prod_{R\in E_{\mathfrak{b}}} \Theta_{E,\mathfrak{a}}(P+R)\right) = \sum_{R\in E_{\mathfrak{b}}} 12N\mathfrak{a}[R] - 12\sum_{Q\in E_{\mathfrak{ab}}}[Q].$$

We conclude by observing that the last equation coincides with the divisor of $\Theta_{E,\mathfrak{a}}(\beta P)$

$$div(\Theta_{E,\mathfrak{a}}(\beta P)) = \sum_{R\in E_{\mathfrak{b}}} 12N\mathfrak{a}[R] - 12\sum_{Q\in E_{\mathfrak{ab}}}[Q].$$

Thus their ratio is a constant $\lambda \in K^{\times}$, and we need to show $\lambda = 1$. Let $\omega_K = \#(\mathcal{O}_K)^{\times}$ and fix a generator α of $\mathfrak{a}$. Using the definition in (3.7) we can evaluate the ratio at $P = O$

$$\begin{aligned}\lambda &= \frac{\prod_{R\in E_{\mathfrak{b}}}\Theta_{E,\mathfrak{a}}(P+R)}{\Theta_{E,\mathfrak{a}}(\beta P)}|_{P=O} = \\ &= \frac{\alpha^{-12N\mathfrak{b}}\Delta(E)^{(N\mathfrak{a}-1)N\mathfrak{b}}\prod_{R\in E_{\mathfrak{b}}}\prod_{Q\in E_{\mathfrak{a}}-\{O\}}(x(P+R)-x(Q))^{-6}}{\alpha^{-12}\Delta(E)^{N\mathfrak{a}-1}\prod_{Q\in E_{\mathfrak{a}}-\{O\}}(x(\beta P)-x(Q))^{-6}}|_{P=O} = \\ &= \frac{\Delta(E)^{(N\mathfrak{a}-1)(N\mathfrak{b}-1)}}{\alpha^{12(N\mathfrak{b}-1)}\beta^{12(N\mathfrak{a}-1)}}\prod_{R\in E_{\mathfrak{b}}-\{O\}}\prod_{Q\in E_{\mathfrak{a}}-\{O\}}(x(R)-x(Q))^{-6} = \mu^{\omega_K}\end{aligned}$$

where

$$\mu = \frac{\Delta(E)^{(N\mathfrak{a}-1)(N\mathfrak{b}-1)/\omega_K}}{\alpha^{12(N\mathfrak{b}-1)/\omega_K}\beta^{12(N\mathfrak{a}-1)/\omega_K}}\prod_{R\in E_{\mathfrak{b}}-\{O\}}\prod_{Q\in E_{\mathfrak{a}}-\{O\}}(x(R)-x(Q))^{-6/\omega_K} = \mu^{\omega_K}.$$

From the fact that $(\mathfrak{a}, 6) = 1$ we then deduce that $N\mathfrak{a} - 1$ is divisible by ω_K. Since ω_K divides 12, then all of the exponents in the definition of μ are integers. Furthermore, μ is fixed by $Gal(\overline{K}/K)$ and then we deduce $\mu \in K^\times$. Let $\mathfrak{q}$ a prime ideal of K, then we have

$$\omega_K v_\mathfrak{q}(\mu) = -12(N\mathfrak{b} - 1)v_\mathfrak{q}(\mathfrak{a}) - 12(N\mathfrak{a} - 1)v_\mathfrak{q}(\mathfrak{b}) - 6 \sum_{R \in E_\mathfrak{b} - O} \sum_{Q \in E_\mathfrak{a} - O} v_\mathfrak{q}(x(R) - x(Q)).$$

Since $\mathfrak{a}, \mathfrak{b}$ are coprime we have that $\mathfrak{q}$ divides only one or neither of them. By Lemma 2.2.2, we then have

$$v_\mathfrak{q}(x(R) - x(Q)) = \begin{cases} -2/(N\mathfrak{q}^m - N\mathfrak{q}^{m-1}) & \text{if } R \text{ has order exactly } \mathfrak{q}^m, m > 0 \\ -2/(N\mathfrak{q}^m - N\mathfrak{q}^{m-1}) & \text{if } Q \text{ has order exactly } \mathfrak{q}^m, m > 0 \\ 0 & \text{if the order of } P \text{ and } Q \text{ is not a power of } \mathfrak{q}. \end{cases}$$

Analogously to the computations in the proof of Theorem 3.1, we deduce $v_\mathfrak{q}(\mu) = 0$ for every prime $\mathfrak{q}$ and then μ is a unit. We conclude $\lambda = \mu^{\omega_K} = 1$. □

Corollary 3.2.1. *Let $\mathfrak{b}$ an integral ideal of K coprime to $\mathfrak{a}$, $Q \in E_\mathfrak{b}$ be a primitive $\mathfrak{b}$-torsion point, $\mathfrak{p}$ a prime ideal dividing $\mathfrak{b}$, π be a generator of $\mathfrak{p}$, and $\mathfrak{b}' = \mathfrak{b}/\mathfrak{p}$. Let $e = \omega_{\mathfrak{b}'}/\omega_\mathfrak{b}$ then*

$$N_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q)^e = \begin{cases} \Theta_{E,\mathfrak{a}}(\pi Q) & \text{if } \mathfrak{p} | \mathfrak{b}' \\ \Theta_{E,\mathfrak{a}}(\pi Q)^{1-\sigma_\mathfrak{p}^{-1}} & \text{if } \mathfrak{p} \nmid \mathfrak{b}' \end{cases}$$

where in the latter case $\sigma_\mathfrak{p} = (\mathfrak{p}, K(\mathfrak{b}')/K)$ is the Frobenius of $\mathfrak{p}$ in $Gal(K(\mathfrak{b}')/K)$.

Proof. Let U denote the multiplicative group $U = 1 + \mathfrak{b}'(O/\mathfrak{b})$. By Theorem A.4 we have the following diagram

$$\begin{array}{ccccccccc} & & & & U & \longrightarrow & Gal(K(\mathfrak{b})/K(\mathfrak{b}')) & \longrightarrow & 1 & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}_K^\times/(\mathcal{O}_K^\times \cap K^{\mathfrak{b},1}) & \longrightarrow & K^\mathfrak{b}/K^{\mathfrak{b},1} & \longrightarrow & Cl_K^\mathfrak{b} & \longrightarrow & Cl_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{O}_K^\times/(\mathcal{O}_K^\times \cap K^{\mathfrak{b}',1}) & \longrightarrow & K^{\mathfrak{b}'}/K^{\mathfrak{b}',1} & \longrightarrow & Cl_K^{\mathfrak{b}'} & \longrightarrow & Cl_K & \longrightarrow & 1 \end{array}$$

where $K^\mathfrak{b}/K^{\mathfrak{b},1} \cong (\mathcal{O}_K/\mathfrak{b})^\times$ and $K^{\mathfrak{b}'}/K^{\mathfrak{b}',1} \cong (\mathcal{O}_K/\mathfrak{b}')^\times$. Applying the Snake Lemma we get the following map is surjective

$$U \twoheadrightarrow Gal(K(\mathfrak{b})/K(\mathfrak{b}'))$$

denote by $u \mapsto \sigma_u$ with kernel of cardinality $e = \omega_{\mathfrak{b}'}/\omega_\mathfrak{b}$. Therefore, by Theorem 3.1.(ii) we have

$$N_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q)^e = \prod_{u \in U} \Theta_{E,\mathfrak{a}}^{\sigma_u} = \prod_{u \in U} \Theta_{E,\mathfrak{a}}(uQ).$$

We then now observe

$$\{uQ : u \in U\} = \{P \in E[\mathfrak{b}] : \pi P = \pi Q,\ P \notin E[\mathfrak{b}']\} =$$
$$= \begin{cases} \{Q + R : R \in E_\mathfrak{p}\} & \text{if } \mathfrak{p} | \mathfrak{b}' \\ \{Q + R : R \in E_\mathfrak{p},\ R \not\equiv -Q (\text{mod } E_{\mathfrak{b}'})\} & \text{if } \mathfrak{p} \nmid \mathfrak{b}'. \end{cases}$$

Thus by Distribution relation Theorem 3.2, if $\mathfrak{p}|\mathfrak{b}'$ we have

$$N_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q)^e = \prod_{R\in E_{\mathfrak{p}}} \Theta_{E,\mathfrak{a}}(Q+R) = \Theta_{E,\mathfrak{a}}(\pi Q).$$

Similarly, if $\mathfrak{p} \nmid \mathfrak{b}'$

$$\Theta_{E,\mathfrak{a}}(Q+R_0)N_{K(\mathfrak{b})/K(\mathfrak{b}')}\Theta_{E,\mathfrak{a}}(Q)^e = \Theta_{E,\mathfrak{a}}(\pi Q)$$

where $R_0 \in E_{\mathfrak{p}}$ satisfies $Q + R_0 \in E_{\mathfrak{b}'}$. By Theorem 3.1.(ii) we have

$$\Theta_{E,\mathfrak{a}}(Q+R_0)^{\sigma_{\mathfrak{p}}} = \Theta_{E,\mathfrak{a}}(\pi Q + \pi R_0) = \Theta_{E,\mathfrak{a}}(\pi Q)$$

so this completes the proof. □

3.1.2 Functions on complex lattices

In order to study the theta function from the complex analytical point of view we recall the definition and properties of some classical complex functions.

Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$, whose basis is ordered so that $\tau = \omega_1/\omega_2$ belongs to the upper half-plane.

Definition 3.2. *Let $\sigma(z, L)$ be the Weierstrass's σ-function and Ramanujans's Δ function defined by the absolutely convergent products*

$$\sigma(z,L) = z \prod_{\omega\in L-\{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right)$$
$$\Delta(L) = \left(\frac{2\pi}{\omega_2}\right)^{12} q_\tau \prod_{n=1}^{\infty}(1-q_\tau^n)^{24}, \quad q_\tau = \exp(2\pi i\tau).$$

Definition 3.3. *For a lattice L define*

$$A(L) := \pi^{-1} \mathit{area}(\mathbb{C}/L) = (2\pi i)^{-1}(\omega_1\overline{\omega_2} - \omega_2\overline{\omega_1})$$
$$s_2(L) := \lim_{s\to 0^+} \sum_{\omega\in L-\{0\}} \omega^{-2}|\omega|^{-2s}$$
$$\eta(z,L) := A(L)^{-1}\overline{z} + s_2(L)z$$

In particular, the Weierstrass's σ-function of L satisfies the following transformation law.

Lemma 3.1.2 (Transformation law). *Let $\omega \in L$, then we have the following identity*

$$\sigma(z+\omega, L) = \pm\sigma(z,L)\exp\left(\eta(\omega,L)\left(z+\frac{\omega}{2}\right)\right) \tag{3.4}$$

Proof. See [We] Chapter IV,3-4. □

Definition 3.4. *Define the fundamental theta function θ as*

$$\theta(z,L) := \Delta(L)\exp(-6\eta(z,L)z)\sigma(z,L)^{12}. \tag{3.5}$$

The theta function is non-holomorphic due to $\overline{z}$ in $\eta(z, L)$.

We have the following useful homothetic relation.

Lemma 3.1.3. *For every $c \in \mathbb{C}$ non-zero we have*

$$\theta(cz, cL) = \theta(z, L). \tag{3.6}$$

Proof. By definition we have

$$\eta(cz, cL) = A(cL)^{-1}\overline{cz} + s_2(cL)cz = c^{-1}\eta(z, L)$$

and

$$\sigma(cz, cL) = cz \prod_{\omega \in L - \{0\}} \left(1 - \frac{cz}{c\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\left(c\frac{z}{c\omega}\right)^2\right) = c\sigma(z, L).$$

We conclude

$$\begin{aligned}\theta(cz, cL) &= \Delta(cL)\exp(-6\eta(cz, cL)cz)\sigma(cz, cL)^{12} = \\ &= c^{-12}\Delta(L)\exp(-6\eta(z, L)z)c^{12}\sigma(z, L)^{12} = \theta(z, L).\end{aligned}$$

□

3.1.3 Theta functions over $\mathbb{C}$

Let $\mathfrak{a} \subseteq \mathcal{O}_K$ be an integral ideal of K prime to the ideals in S. We denote $N\mathfrak{a}$ the absolute norm of $\mathfrak{a}$ and $\mathfrak{a}^{-1}L$ denotes the lattice $\Omega_\infty \mathfrak{a}^{-1}$.

Definition 3.5. *Define $\Theta(z, \mathfrak{a})$ as an elliptic function*

$$\Theta(z, \mathfrak{a}) = \alpha^{-12}\Delta(L)^{N\mathfrak{a}-1} \prod_{u \in \mathfrak{a}^{-1}L/L - \{0\}} (\wp(z) - \wp(u))^{-6} \tag{3.7}$$

where the product is taken over a set of representatives $\{u\}$ of the non-zero cosets of $\mathfrak{a}^{-1}L/L$. Recall that the number of cosets $\mathfrak{a}^{-1}L/L$ is $N\mathfrak{a}$.

From this equation, it follows $\Theta(z, \mathfrak{a}) = \Theta_{E,\mathfrak{a}} \circ \xi$. In particular, we can restate the results of Section 3.1.1 in terms of complex L-elliptic functions.

Theorem 3.3. *Let $\mathfrak{b}$ a nontrivial ideal of $\mathcal{O}_K$ relatively prime to $\mathfrak{a}$, let $v \in \mathfrak{b}^{-1}L$ be a primitive $\mathfrak{b}$-division point.*

(i) $\Theta(v, \mathfrak{a}) \in K(\mathfrak{b})$.

(ii) *If $\mathfrak{c}$ is an ideal of $\mathcal{O}_K$ prime to $\mathfrak{b}$, c a generator of $\mathfrak{c}$, and $\sigma_\mathfrak{c} = (\mathfrak{c}, K(\mathfrak{b})/K)$ its Artin symbol. Then we have*

$$\Theta(v, \mathfrak{a})^{\sigma_\mathfrak{c}} = \Theta(cv, \mathfrak{a})$$

(iii) *If $\mathfrak{b}$ is not a prime power then $\Theta(v, \mathfrak{a})$ is a global unit of $K(\mathfrak{b})$. If $\mathfrak{b}$ is a power of $\mathfrak{p}$ then $\Theta(v, \mathfrak{a})$ is a unit at primes not dividing $\mathfrak{p}$.*

Theorem 3.4 (Distribution relation)**.** *Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals of K, relatively prime to each other, let $\beta \in \mathcal{O}_K$ any generator of $\mathfrak{b}$. Then we have*

$$\prod_{v \in \mathfrak{b}^{-1}L/L} \Theta(z+v, \mathfrak{a}) = \Theta(\beta z, \mathfrak{a})$$

where the product is taken over the set of representatives of $\mathfrak{b}$-division points.

Prop. 3.1.1. $\Theta(z, \mathfrak{a}, L) = \theta(z, L)^{N\mathfrak{a}}/\theta(z, \mathfrak{a}^{-1}L)$.

Proof. Let $f(z) = \theta(z, L)^{N\mathfrak{a}}/\theta(z, \mathfrak{a}^{-1}L)$ then we have that $f(z)$ is meromorphic. Indeed, we have explicitly

$$f(z) = \frac{\Delta(L)^{N\mathfrak{a}}\sigma(z, L)^{12N\mathfrak{a}}}{\Delta(\mathfrak{a}^{-1}L)\sigma(z, \mathfrak{a}^{-1}L)^{12}} \exp[-6z(N\mathfrak{a}\eta(z, L) - \eta(z, \mathfrak{a}^{-1}L))].$$

In particular, we have that the factor containing $\overline{z}$ cancels out

$$\begin{aligned} N\mathfrak{a}\eta(z, L) - \eta(z, \mathfrak{a}^{-1}L) &= N\mathfrak{a}A(L)^{-1}\overline{z} + N\mathfrak{a}s_2(L)z - A(\mathfrak{a}^{-1}L)^{-1}\overline{z} - s_2(\mathfrak{a}^{-1}L)z = \\ &= N\mathfrak{a}s_2(L)z - s_2(\mathfrak{a}^{-1}L)z. \end{aligned}$$

We conclude $f(z)$ is meromorphic. Furthermore, $f(z)$ is periodic with respect to L. In fact, using the transformation law 3.1.2 we have for every $\omega \in L$

$$\begin{aligned} \frac{\sigma(z+\omega, L)^{12N\mathfrak{a}}}{\sigma(z+\omega, \mathfrak{a}^{-1}L)^{12}} &= \frac{\sigma(z, L)^{12N\mathfrak{a}} \exp\left(12N\mathfrak{a}\eta(\omega, L)\left(z + \frac{\omega}{2}\right)\right)}{\sigma(z, \mathfrak{a}^{-1}L) \exp\left(\eta(\omega, \mathfrak{a}^{-1}L)\left(z + \frac{\omega}{2}\right)\right)} = \\ &= \frac{\sigma(z, L)^{12N\mathfrak{a}}}{\sigma(z, \mathfrak{a}^{-1}L)} \exp\left(\left(z + \frac{\omega}{2}\right)\left(12N\mathfrak{a}\eta(\omega, L) - \eta(\omega, \mathfrak{a}^{-1}L)\right)\right) \end{aligned}$$

and

$$\eta(z+\omega, L) = A(L)^{-1}\overline{(z+\omega)} + s_2(L)(z+\omega) = \eta(z, L) + \eta(\omega, L).$$

Then we conclude $f(z+\omega) = f(z)$. From the explicit definition of σ we have that the divisor of $\theta(z, L)$ is given by

$$12N\mathfrak{a}[0] - \sum_{u \in \mathfrak{a}^{-1}L/L} [u].$$

By Lemma 3.1.1 we deduce $\Theta(z, L) = \lambda f(z)$ with $\lambda \in \mathbb{C}^\times$ constant. Both functions have Laurent series beginning with $\alpha^{-12}\Delta(L)^{12(N\mathfrak{a}-1)}z^{12(N\mathfrak{a}-1)}$, so $\lambda = 1$. □

Corollary 3.4.1. *Let $\mathfrak{b}$ a nontrivial ideal of $\mathcal{O}_K$ relatively prime to $\mathfrak{a}$, let $v \in \mathfrak{b}^{-1}L$ be a primitive $\mathfrak{b}$-division point. If $\mathfrak{c}$ is an ideal of $\mathcal{O}_K$ prime to $\mathfrak{b}$, and $\sigma_\mathfrak{c} = (\mathfrak{c}, K(\mathfrak{b})/K)$ its Artin symbol. Then we have*

$$\Theta(v, \mathfrak{a})^{\sigma_\mathfrak{c}} = \Theta(v, \mathfrak{a}\mathfrak{c})\Theta(v, \mathfrak{c})^{-N\mathfrak{a}}.$$

Proof. Recall that by Theorem 3.3 we have

$$\Theta(v, \mathfrak{a})^{\sigma_\mathfrak{c}} = \Theta(cv, \mathfrak{a})$$

with c a generator of $\mathfrak{c}$. Then by the previous proposition, we get

$$\Theta(v, \mathfrak{a})^{\sigma_\mathfrak{c}} = \frac{\theta(cz, L)^{N\mathfrak{a}}}{\theta(cz, \mathfrak{a}^{-1}L)} = \frac{\theta(z, \mathfrak{c}L)^{N\mathfrak{a}}}{\theta(z, \mathfrak{c}^{-1}\mathfrak{a}^{-1}L)}$$

where the last equality follows from Lemma 3.1.3. Then we conclude

$$\frac{\theta(z,\mathfrak{c}L)^{N\mathfrak{a}}}{\theta(z,\mathfrak{c}^{-1}\mathfrak{a}^{-1}L)} = \frac{\theta(z,\mathfrak{c}^{-1}L)^{N\mathfrak{a}}}{\theta(z,\mathfrak{c}\mathfrak{a}^{-1}L)} \cdot \frac{\theta(z,L)^{N\mathfrak{a}\mathfrak{c}}}{\theta(z,L)^{N\mathfrak{a}\mathfrak{c}}} = \Theta(z,\mathfrak{a}\mathfrak{c})\Theta(z,\mathfrak{c})^{-N\mathfrak{a}}.$$

□

3.2 Eisenstein series and L-functions

3.2.1 Eisenstein numbers

Let L be a lattice in the complex plane generated by ω_1 and ω_2 with $\Im(\omega_2/\omega_1) > 0$. Recall that we have $A(L) = (2\pi i)^{-1}(\omega_2\overline{\omega}_1 - \omega_1\overline{\omega}_2)$ is the area of the lattice L.

Definition 3.6. *We define a pairing for $z, w \in \mathbb{C}$ by*

$$\langle z, w\rangle_L = \exp\left(\frac{z\overline{w} - w\overline{z}}{A(L)}\right)$$

Lemma 3.2.1. *For $a, z, w \in \mathbb{C}$ the pairing has the following properties.*

(i) $\langle z, w\rangle_L = \langle -w, z\rangle_L = \langle w, z\rangle_L^{-1}$,

(ii) $\langle az, w\rangle_L = \langle z, \overline{a}w\rangle_L$,

(iii) *$z \in L$ if and only if $\langle z, \omega\rangle_L = 1$ for all $\omega \in L$.*

Definition 3.7. *For each integer $k \geq 1$, $z \in \mathbb{C} - L$, and $w \in \mathbb{C}$, we define the Eisenstein-Kronecker-Lerch series $H_k(z, w, s, L)$ by*

$$H_k(z, w, s, L) = \sum_{\gamma \in L} \frac{(\overline{z} + \overline{\gamma})^k}{|z + \gamma|^{2s}}\langle \gamma, w\rangle_L, \qquad \Re s > k/2 + 1.$$

For a fixed $z_0, w_0 \in \mathbb{C}$, we define $H_k^*(z_0, w_0, s, L)$ by

$$H_k^*(z_0, w_0, s, L) = \sum_{\gamma \in L}^{*} \frac{(\overline{z_0} + \overline{\gamma})^k}{|z_0 + \gamma|^{2s}}\langle \gamma, w_0\rangle_L, \qquad \Re s > k/2 + 1.$$

where the sum is take over all $\omega \in L$ other than $-z_0$ if $z_0 \in L$. The series converges absolutely for $\Re s > k/2 + 1$

Observe that $H_k(z, w, s, L) = H_k^*(z, w, s, L)$ if $z \in \mathbb{C} - L$.

Prop. 3.2.1. *Let $k \geq 1$ be an integer.*

(i) *The function $\Gamma(s)H_k^*(z, w, s, L)$ for s continues meromorphically to a function on the whole complex plane, with possible poles only at*

(a) *$k = 0$, $z \in L$ with simple pole at $s = 0$ and residue $\langle -z, w\rangle$;*

(b) *$k = 0$, $w \in L$ with simple pole at $s = 1$ and residue $A(L)^{-1}$ if $w = 0$.*

(ii) $H_k^*(z,w,s,L)$ *satisfies the functional equation*

$$\Gamma(s)H_k^*(z,w,s,L) = A(L)^{k+1-2s}\Gamma(k+1-s)H_k^*(w,z,k+1-s,L)\langle w,z\rangle_L.$$

Proof. See [Wei76], VIII, 12. □

The function $z \mapsto H_k^*(z,w,s,L)$ is even or odd if k is even or odd. Furthermore, it is periodic of period L. Let $\mathcal{D}$ the differential operator defined by

$$\mathcal{D} = \overline{z}\frac{\partial}{\partial z} + \overline{\omega}_1\frac{\partial}{\partial \omega_1} + \overline{\omega}_2\frac{\partial}{\partial \omega_2}.$$

Observe that in particular, we have $\mathcal{D}(A(L)) = 0$. The Eisenstein-Kronecker-Lerch series for various integers k and s are related by the following differential equations.

Lemma 3.2.2. *Let $k > 0$ be an integer. The function $H_k(z,w,s)$ satisfied the differential equations*

$$\begin{aligned}
\partial_z H_k(z,w,s) &= -sH_{k+1}(z,w,s+1),\\
\partial_{\overline{z}} H_k(z,w,s) &= (k-s)H_{k-1}(z,w,s),\\
\partial_w H_k(z,w,s) &= -A^{-1}(H_{k+1}(z,w,s) - \overline{z}H_k(z,w,s)),\\
\partial_{\overline{w}} H_k(z,w,s) &= A^{-1}(H_{k-1}(z,w,s-1) - zH_k(z,w,s)),\\
\mathcal{D}H_k(x,0,s) &= -sH_{k+2}(x,0,s+1)
\end{aligned}$$

Proof. The proof follows by the definition of the Eisenstein-Kronecker-Lerch series for $\Re s > a/2+1$. The statement for general s is obtained by analytic continuation. □

Definition 3.8. *Let $z_0, w_0 \in \mathbb{C}$. For any integer $k > 0$, $j \geq 0$, we define the Eisenstein-Kronecker number $E_{j,k}(z_0,w_0,L)$ by*

$$\begin{aligned}
E_{j,k}(z_0,w_0,L) &= H_{j+k}^*(z_0,w_0,k,L) = \\
&= \sum_{\gamma\in L}\frac{(\overline{z}_0+\overline{w}_0)^j}{(z_0+\gamma)^k}\langle\gamma,w_0\rangle \quad \text{if } k \geq j+3.
\end{aligned}$$

For simplicity we will omit the variable w when we will consider $w \in L$, $E_{j,k}(z,L) = E_{j,k}(z,0,L)$.

Definition 3.9. *For $k \geq 1$ define the Eisenstein series by*

$$\begin{aligned}
E_k(z,L) &= H_k^*(z,0,k,L) = \\
&= \sum_{\gamma\in L}\frac{1}{(z+\gamma)^k} \quad \text{if } k \geq 3.
\end{aligned}$$

In particular we have $E_k(z,L) = E_{0,k}(z,L)$.

By definition, we have the following homogeneity properties.

Lemma 3.2.3. *For every $\lambda \in \mathbb{C}^\times$ and every $k > j \geq 0$ we have*

(i) $E_k(\lambda z, \lambda L) = \lambda^{-k}E_k(z,L)$,

(ii) $E_{j,k}(\lambda z,\lambda L) = \dfrac{\overline{\lambda}^{j+k}}{|\lambda|^{2k}}E_{j,k}(z,L)$.

Lemma 3.2.4. (i) $E_1(z,L) = \frac{d}{dz}\log(\sigma(z,L)) - s_2(L)z - A(L)^{-1}\overline{z}$,

(ii) $E_2(z,L) = -\frac{d}{dz}E_1(z,L) = \wp(z,L) + s_2(L)$,

(iii) $E_k(z,L) = \frac{(-1)^k}{(k-1)!}\left(\frac{d}{dz}\right)^{k-2}\wp(z,L)$ *if* $k \geq 3$.

(iv) $E_{j,k}(z,L) = \frac{(-1)^{k-1}}{(k-1)!}\mathcal{D}^j\partial_z^{k-j-1}E_1(z,L)$ *for all* $k > j \geq 0$.

Proof. See [Wei76]. □

Prop. 3.2.2. *For $k > j \geq 0$ integers, exists a polynomial $P_{j,k}$ in $j+k$ indeterminates, of degree $j+1$, with integer coefficients such that*

$$E_{j,k}(z,L) = \frac{(A(L)/2)^j}{(k-1)(k-2)\cdots(k-j)}P_{j,k}(E_1(z,L),\ldots,E_{j+k}(z,L))$$

Proof. See [Wei76].VI.4. □

3.2.2 Relation with L-values

Definition 3.10. *Define the Hecke L-functions associated to powers of $\overline{\psi}$ to be the analytic continuations of the Dirichlet series*

$$L(\overline{\psi}^k, s) = \sum_{\mathfrak{b}}\frac{\overline{\psi}^k(\mathfrak{b})}{N\mathfrak{b}^s}$$

summing over integral ideals $\mathfrak{b}$ of K prime to the conductor of $\overline{\psi}^k$. If $\mathfrak{m}$ is an integral ideal of K divisible by $\mathfrak{f}$ and $\sigma \in Gal(K(\mathfrak{m})/K)$ then we define the partial Dirichlet series

$$L_{K(\mathfrak{m})}(\overline{\psi}^k, s, \sigma) = \sum_{(\mathfrak{b},K(\mathfrak{m})/K)=\sigma}\frac{\overline{\psi}^k(\mathfrak{b})}{N\mathfrak{b}^s}$$

where the sum is restricted to the integral ideals of K prime to $\mathfrak{m}$ whose Artin symbol for the extension $K(\mathfrak{m})/K$ is σ.

Theorem 3.5. *Let $\mathfrak{m}$ be an integral ideal of K divisible by $\mathfrak{f}$, and consider ρ a $\mathfrak{m}$-division point. Then for every $k > j \geq 0$,*

$$E_k(\rho,L) = \rho^{-k}\psi(\mathfrak{c})^k L_{K(\mathfrak{m})}(\overline{\psi}^k, k, \mathfrak{c}), \tag{3.8}$$

$$E_{j,k}(\rho,L) = \rho^{-j-k}Nm^{-j}|\Omega_\infty|^{2j}\psi(\mathfrak{c})^{j+k}L_{K(\mathfrak{m})}(\overline{\psi}^{j+k}, k, \mathfrak{c}). \tag{3.9}$$

where $\mathfrak{c} = \Omega_\infty^{-1}\rho\mathfrak{m}$.

Proof. Let $m \in \mathcal{O}$ be a generator of $\mathfrak{m}$, then we have $\rho = \alpha\Omega_\infty/m$ for some $\alpha \in \mathcal{O}$ prime to $\mathfrak{m}$. Then we have for $\Re s > k/2+1$

$$H_k^*(\rho,0,s,L) = \sum_{\omega\in L}\frac{(\overline{\rho}+\overline{w})^k}{|\rho+w|^{2s}} = \sum_{\beta\in\mathcal{O}}\frac{(\overline{\alpha}\overline{\Omega_\infty}/\overline{m}+\overline{\Omega_\infty\beta})^k}{|\alpha\Omega_\infty/m+\beta\Omega_\infty|^{2s}} =$$

$$= \frac{Nm^s}{\overline{m}^k}\frac{\overline{\Omega_\infty}^k}{|\Omega_\infty|^{2s}}\sum_{\substack{\beta\in\mathcal{O},\\ \beta\equiv\alpha \mod \mathfrak{m}}}\frac{\overline{\beta}^k}{|\beta|^{2s}}.$$

By definition of the Hecke character ψ we can define

$$\varepsilon(\beta) = \psi(\beta\mathcal{O})/\beta$$

to be a multiplicative map from $\{\beta \in \mathcal{O} : (\beta\mathcal{O}, \mathfrak{f}) = 1\}$ to $\mathcal{O}^\times$. By definition of the conductor, ε factors through $(\mathcal{O}/\mathfrak{f})^\times$. Thus if $\beta \equiv \alpha \mod \mathfrak{m}$, we have $\varepsilon(\beta) = \varepsilon(\alpha)$ and then

$$\overline{\beta} = \overline{\psi}(\beta\mathcal{O})\frac{\psi(\alpha\mathcal{O})}{\alpha}.$$

Therefore we can rewrite

$$\sum_{\substack{\beta\in\mathcal{O}, \\ \beta\equiv\alpha \mod \mathfrak{m}}} \frac{\overline{\beta}^k}{|\beta|^{2s}} = \frac{\psi(\alpha\mathcal{O})^k}{\alpha^k} \sum_{(\mathfrak{b},K(\mathfrak{m})/K)=(\mathfrak{a},K(\mathfrak{m})/K)} \frac{\overline{\psi}^k(\mathfrak{b})}{N\mathfrak{b}^s} =$$

$$= \frac{\psi(\mathfrak{c})^k}{\alpha^k} L_{K(\mathfrak{m})}(\overline{\psi}^k, s, \mathfrak{c})$$

where $\mathfrak{c} = \alpha\mathcal{O} = \rho\mathfrak{m}\Omega^{-1}$. In particular, we then conclude

$$E_k(\rho, L) = H_k^*(\rho, 0, k, L) = \frac{m^k}{\alpha^k \Omega_\infty^k}\psi(\mathfrak{c})L_{K(\mathfrak{m})}(\overline{\psi}^k, k, \mathfrak{c})$$

where we recall $\rho = \alpha\Omega_\infty/m$. Analogously we have

$$E_{j,k}(\rho, L) = H_{j+k}^*(\rho, 0, k, L) = \frac{Nm^k}{\overline{m}^{j+k}} \frac{\overline{\Omega_\infty}^{j+k}}{|\Omega_\infty|^{2k}} \frac{\psi(\mathfrak{c})^{j+k}}{\alpha^{j+k}} L_{K(\mathfrak{m})}(\overline{\psi}^{j+k}, k, \mathfrak{c})$$

$$= \rho^{-j-k} Nm^{-j} |\Omega_\infty|^{2j} \psi(\mathfrak{c})^{j+k} L_{K(\mathfrak{m})}(\overline{\psi}^{j+k}, k, \mathfrak{c}).$$

□

Corollary 3.5.1. *Let $\mathfrak{m}$ be an integral ideal of K divisible by $\mathfrak{f}$, and consider ρ a $\mathfrak{m}$-division point. Let B a set of ideals of $\mathcal{O}_K$, prime to $\mathfrak{m}$ such that the Artin map $\mathfrak{b} \mapsto (\mathfrak{b}, K(\mathfrak{m})/K)$ is a bijection from B to $Gal(K(\mathfrak{m})/K)$. Then for every $k > j \geq 0$,*

$$\sum_{\mathfrak{b}\in B} E_k(\psi(\mathfrak{b})\rho, L) = \rho^{-k} L_{K(\mathfrak{m})}(\overline{\psi}^k, k),$$

$$\sum_{\mathfrak{b}\in B} E_{j,k}(\psi(\mathfrak{b})\rho, L) = \rho^{-(j+k)} N\mathfrak{m}^{-j} |\Omega_\infty|^{2j} L_{K(\mathfrak{m})}(\overline{\psi}^{j+k}, k)$$

Proof. Applying the previous theorem

$$\sum_{\mathfrak{b}\in B} E_k(\psi(\mathfrak{b})\rho, L) = \rho^{-k} \sum_{\mathfrak{b}\in B} \psi(\mathfrak{b})^{-k}\psi(\mathfrak{c}_\mathfrak{b})^k L_{K(\mathfrak{m})}(\overline{\psi}^k, k, \mathfrak{c}_\mathfrak{b})$$

where $\mathfrak{c}_\mathfrak{b} = \Omega_\infty^{-1}\psi(\mathfrak{b})\rho\mathfrak{m}$. Since ρ is a $\mathfrak{m}$-division point, we have $\mathcal{O} = \Omega_\infty^{-1}\rho\mathfrak{m}$ and then $\mathfrak{c}_\mathfrak{b} = \mathfrak{b}$. We can then rewrite

$$\sum_{\mathfrak{b}\in B} E_k(\psi(\mathfrak{b})\rho, L) = \rho^{-k} \sum_{\mathfrak{b}\in B} L_{K(\mathfrak{m})}(\overline{\psi}^k, k, \mathfrak{b}) = \rho^{-k} L_{K(\mathfrak{m})}(\overline{\psi}^k, k).$$

Analogously we have

$$\sum_{\mathfrak{b}\in B} E_{j,k}(\psi(\mathfrak{b})\rho, L) = \sum_{\mathfrak{b}\in B} \psi(\mathfrak{b})^{-j-k}\rho^{-j-k} Nm^{-j} |\Omega_\infty|^{2j} \psi(\mathfrak{c}_\mathfrak{b})^{j+k} L_{K(\mathfrak{m})}(\overline{\psi}^{j+k}, k, \mathfrak{c}_\mathfrak{b}) =$$

$$= \rho^{-j-k} Nm^{-j} |\Omega_\infty|^{2j} L_{K(\mathfrak{m})}(\overline{\psi}^{j+k}, k)$$

□

Theorem 3.6. *For all integral ideal $\mathfrak{a}$ of K, we have that the Laurent expansion of* $\log\Theta(z,\mathfrak{a})$ *at* $z_0\in\mathbb{C}$ *is given by*

$$\frac{d}{dz}\log\Theta(z,\mathfrak{a}) = 12\sum_{k\geq 1}(-1)^k(N\mathfrak{a}E_k(z_0,L)-E_k(z_0,\mathfrak{a}^{-1}L))(z-z_0)^{k-1}.$$

Proof. First of all observe that from the definition 3.4 of $\theta(z,L)$ we have

$$\log(\theta(z,L)) = \log(\Delta(L)) - 6s_2(L)z^2 - 6A(L)^{-1}z\overline{z} + 12\log(\sigma(z,L)),$$
$$\frac{d}{dz}\log(\theta(z,L)) = -12s_2(L)z - 6A(L)^{-1}\overline{z} + 12\frac{d}{dz}\log(\sigma(z,L))$$

Then from Prop. 3.1.1 we get

$$\begin{aligned}\frac{d}{dz}\log\Theta(z,\mathfrak{a}) =& N\mathfrak{a}\frac{d}{dz}\log\theta(z,L) - \frac{d}{dz}\log\theta(z,\mathfrak{a}^{-1}L) =\\ =& -N\mathfrak{a}\left(12s_2(L)z + 6A(L)^{-1}\overline{z} - 12\frac{d}{dz}\log(\sigma(z,L))\right) +\\ &+ 12s_2(\mathfrak{a}^{-1}L)z + 6A(\mathfrak{a}^{-1}L)^{-1}\overline{z} - 12\frac{d}{dz}\log(\sigma(z,\mathfrak{a}^{-1}L)).\end{aligned}$$

Since $A(\mathfrak{a}^{-1}L) = A(L)N\mathfrak{a}$ then using the Lemma 3.2.4 we get that the right-hand side of the previous equation is equal to

$$\frac{d}{dz}\log\Theta(z,\mathfrak{a}) = 12N\mathfrak{a}E_1(z,L) - 12E_1(z,\mathfrak{a}^{-1}L).$$

By repeated differentiation using Lemma 3.2.4 we obtain

$$\left(\frac{d}{dz}\right)^k\log\Theta(z,\mathfrak{a}) = 12(-1)^k(k-1)!\left(N\mathfrak{a}E_k(z,L) - E_k(z,\mathfrak{a}^{-1}L)\right).$$

□

Corollary 3.6.1. *Let $\mathfrak{m}$ be an integral ideal of K, and consider ρ an $\mathfrak{m}$-division point. Then for every integral ideal $\mathfrak{b}$ of K coprime with $\mathfrak{m}$, the following hold*

(i) $E_k(\rho,L)\in K(E_\mathfrak{m})$,

(ii) $\left(\frac{A(L)}{2}\right)^{-j}E_{j,k}(\rho,L)\in K(E_\mathfrak{m})$

(iii) $E_k(\rho,L)^{\sigma_\mathfrak{b}} = E_k(\psi(\mathfrak{b})\rho,L)$,

where $\sigma_\mathfrak{b}$ is the Artin symbol associated to $\mathfrak{b}$.

Proof. First of all, observe that since $\Theta(z,L)$ is a rational function in $\wp(z,L)$ with coeffiecients in K, then by addition theorem we deduce that $\Theta(z+\rho,L)$ is a rational function in $\wp(z,L)$ and $\wp'(z,L)$ with coefficients in $K(E_\mathfrak{m})$. By the previous theorem we conclude that the coefficients of the Laurent expansion

$$(-1)^k(k-1)!(N\mathfrak{a}E_k(\rho,L) - E_k(z_0,\mathfrak{a}^{-1}L))(z-\rho)^k$$

are in $K(E_{\mathfrak{m}})$. Consider now $\alpha \in \mathcal{O}_K$ such that $\alpha \equiv 1 \mod \mathfrak{m}$ and take $\mathfrak{a} = (\alpha)$. In particular we then have $\alpha\rho \equiv \rho \mod L$. By homogeneity property of the Eisenstein series we have $E_k(\rho, \mathfrak{a}^{-1}L) = \alpha^k E_k(\alpha\rho, L)$ and then we conclude

$$\left(\frac{d}{dz}\right)^k \log\Theta(z, \mathfrak{a}) = (-1)^{k-1}12(k-1)!(N\alpha - \alpha^k)E_k(\rho, L).$$

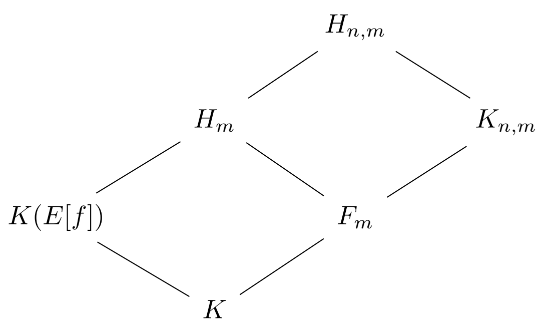
This proves that $E_k(\rho, L)$ belongs to $K(E_{\mathfrak{m}})$. By Prop. 3.2.2 we deduce (ii). Furthermore, considering $\mathfrak{b}$ an integral ideal of K coprime with $\mathfrak{m}$ and applying $\sigma_{\mathfrak{b}}$ to the previous expression we get

$$\begin{aligned}
E_k(\rho, L)^{\sigma_{\mathfrak{b}}} =& \frac{(-1)^k}{12(k-1)!(N\mathfrak{a} - \alpha^k)}\left(\left(\frac{d}{dz}\right)^k \log\Theta(z+\rho, \mathfrak{a})^{\sigma_{\mathfrak{b}}}|_{z=0}\right) = \\
=& \frac{(-1)^k}{12(k-1)!(N\mathfrak{a} - \alpha^k)}\left(\left(\frac{d}{dz}\right)^k \log\Theta(z+\psi(\mathfrak{b})\rho, \mathfrak{a})|_{z=0}\right) = \\
=& E_k(\psi(\mathfrak{b})\rho, L)
\end{aligned}$$

that concludes the proof. □

3.3 Elliptic units

For simplicity, denote $H_m = K\left(\mathfrak{f}\overline{\mathfrak{p}}^{m+1}\right)$ and $H_{n,m} = K\left(\mathfrak{f}\mathfrak{p}^{\mathfrak{n}+1}\overline{\mathfrak{p}}^{m+1}\right)$ the Ray class fields of K respectively modulo $\mathfrak{f}\overline{\mathfrak{p}}^{m+1}$ and $\mathfrak{f}\mathfrak{p}^{\mathfrak{n}+1}\overline{\mathfrak{p}}^{m+1}$. Recall that by Lemma 2.2.4 we have the following diagram of extensions of fields.



In the previous section, we have seen two special properties of Eisenstein numbers. The first is that they are related to values of L series and the second one is that the appear in the Laurent expansion of the logarithm of theta function. In order to construct a measure that interpolates some special L values, we construct a new rational function as product of translated theta function. The exactly relation between the lambda function and the L values is provided in Theorem 3.7.

Definition 3.11. *Let $\rho = \Omega_\infty/f$ and let B be a set of integral ideals of K prime to $\mathfrak{f}$ such that the set of elements $(\mathfrak{b}, K(\mathfrak{f})/K)$ for $\mathfrak{b} \in B$ is a set of representatives for the Galois group $Gal(K(\mathfrak{f})/K)$. For $\mathfrak{a}$ integral ideal of K prime to $6p\mathfrak{f}$, we set*

$$\Lambda(z, \mathfrak{a}) = \prod_{\mathfrak{b}\in B} \Theta(z + \psi(\mathfrak{b})\rho, \mathfrak{a}).$$

We can extend the previous definition to higher ray class fields in the following way.

Definition 3.12. *Let $\rho_m = \Omega_\infty / f\overline{\pi}^{m+1}$ and let B_m be a set of integral ideals of K prime to $\mathfrak{f}\overline{\mathfrak{p}}$ such that the set of elements $(\mathfrak{b}, H_m/K)$ for $\mathfrak{b} \in B_m$ is a set of representatives for the Galois group $Gal(H_m/F_m)$. For $\mathfrak{a}$ integral ideal of K prime to $6p\mathfrak{f}$, we set*

$$\Lambda_m(z, \mathfrak{a}) = \prod_{\mathfrak{b} \in B_m} \Theta(z + \psi(\mathfrak{b})\rho_m, \mathfrak{a}).$$

Lemma 3.3.1. *The functions $\Lambda(z, \mathfrak{a})$ and $\Lambda_m(z, \mathfrak{a})$ are rational functions of $\wp(z)$ and $\wp'(z)$ with coefficients respectively in K and F_m. They are independent of the choice of the set of ideals B and B_m.*

Proof. We prove directly the statement for $\Lambda_m(z, \mathfrak{a})$. First of all observe that ρ_m is a torsion point $\rho_m \in E[\mathfrak{f}\overline{\pi}^{m+1}]$ and then by Lemma 2.2.4 we have $\rho_m \in E(H_m)$. Since $\Theta(z, \mathfrak{a})$ is a rational function of $\wp(z)$ with coefficients in K, we deduce by the addition formula that $\Theta(z + \rho_m, \mathfrak{a})$ is a rational function of $\wp(z)$ and $\wp'(z)$ with coefficients in H_m. If $\mathfrak{b}$ is an integral ideal prime to $\mathfrak{f}\overline{\mathfrak{p}}$, then by Corollary B.3.1.(ii) we have

$$\xi(\rho_m)^{(\mathfrak{b}, H_m/K)} = \xi(\psi(\mathfrak{b})\rho_m).$$

In particular, we obtain the function $\Theta(z + \psi(\mathfrak{b})\rho_m, \mathfrak{a})$ applying $(\mathfrak{b}, H_m/K)$ to the coefficients of $\Theta(z + \rho_m, \mathfrak{a})$. We conclude the function Λ_m is independent of the choice of B_m. ☐

Theorem 3.7. *For all integral ideal $\mathfrak{a}$ of K and $m \geq 0$, we have the following Laurent expansions*

$$\frac{d}{dz} \log \Lambda(z, \mathfrak{a}) = 12 \sum_{k \geq 1} (-1)^{k-1} f^k \Omega_\infty^{-k} (N\mathfrak{a} - \psi(\mathfrak{a})^k) L_{K(\mathfrak{f})}(\overline{\psi}^k, k)(z - z_0)^{k-1},$$

$$\frac{d}{dz} \log \Lambda_m(\overline{\pi}^{-(m+1)} z, \mathfrak{a}) = 12 \sum_{k \geq 1} (-1)^{k-1} f^k \Omega_\infty^{-k} (N\mathfrak{a} L_{F_m}(\overline{\psi}^k, k, 1) - \psi^k(\mathfrak{a}) L_{F_m}(\overline{\psi}^k, k, \mathfrak{a}).$$

Proof. By definition of Λ we have

$$\log \Lambda(z, \mathfrak{a}) = \sum_{\mathfrak{b} \in B} \log \Theta(z + \psi(\mathfrak{b})\rho, \mathfrak{a}).$$

In particular, Theorem 3.6 gives us

$$\left(\frac{d}{dz}\right)^k \log \Lambda(z, \mathfrak{a})|_{z=0} =$$
$$=(-1)^{k-1} 12(k-1)! \sum_{\mathfrak{b} \in B} (N\mathfrak{a} E_k(\psi(\mathfrak{b})\rho, L) - E_k(\psi(\mathfrak{b})\rho, \mathfrak{a}^{-1}L)).$$

Applying the result of Theorem 3.5.1, we obtain

$$\sum_{\mathfrak{b} \in B} N\mathfrak{a} E_k(\psi(\mathfrak{b})\rho, L) = N\mathfrak{a}\rho^{-k} L_{K(\mathfrak{f})}(\overline{\psi}^k, k)$$
$$\sum_{\mathfrak{b} \in B} E_k(\psi(\mathfrak{b})\rho, \mathfrak{a}^{-1}L) = \sum_{\mathfrak{b} \in B} \psi(\mathfrak{a}) E_k(\psi(\mathfrak{a}\mathfrak{b})\rho, L) = \psi(\mathfrak{a})^k \rho_m^{-k} L_{K(\mathfrak{f})}(\overline{\psi}^k, k)$$

where we have used the fact that $\mathfrak{a}$ is coprime to $6p\mathfrak{f}$ and then $\mathfrak{a}\mathfrak{b}$ defines a new set of representatives for the Galois group $Gal(K(\mathfrak{f})/K)$. Combining the two relations we get

$$\left(\frac{d}{dz}\right)^k \log \Lambda(z,\mathfrak{a})|_{z=0} = \\ =(-1)^{k-1}12(k-1)! \sum_{\mathfrak{b}\in B} (N\mathfrak{a}\psi(\mathfrak{a}))\rho^{-k} L_{K(\mathfrak{f})}(\overline{\psi}^k, k).$$

Since $\rho = \Omega_\infty / f$ we conclude

$$\left(\frac{d}{dz}\right)^k \log \Lambda(z,\mathfrak{a})|_{z=0} = 12(-1)^{k-1} f^k \Omega_\infty^{-k}(k-1)!(N\mathfrak{a} - \psi(\mathfrak{a})^k) L_{K(\mathfrak{f})}(\overline{\psi}^k, k)(z-z_0)^k.$$

The proof of the Laurent of Λ_m is identical paying attention that the sum is taken over B_m galois representatives over H_m/F_m. See [Yag82] for details.

□

Corollary 3.7.1 (Damerell's Theorem I)**.** *For every $k \geq 1$,*

$$\Omega_\infty^{-k} L_{K(\mathfrak{f})}(\overline{\psi}^k, k) \in K.$$

Proof. By Lemma 3.8 we have that $\Lambda(z)$ has a series expansion with coefficients in K. Then the first equation follows directly from the previous theorem. □

Corollary 3.7.2 (Damerell's Theorem II)**.** *For every $k > j \geq 0$,*

$$\left(\frac{2\pi}{\sqrt{d_K}}\right)^j \Omega_\infty^{-(k+j)} L_{K(\mathfrak{f})}(\overline{\psi}^{k+j}, k) \in \overline{K}$$

Proof. Let ρ be an $\mathfrak{f}$-division point. By Corollary 3.5.1 we have the following relation between Eisenstein numbers and L values

$$\sum_{\mathfrak{b}\in B} E_{j,k}(\psi(\mathfrak{b})\rho, L) = \rho^{-(j+k)} N\mathfrak{m}^{-j} |\Omega_\infty|^{2j} L_{K(\mathfrak{f})}(\overline{\psi}^{j+k}, k). \tag{3.10}$$

Furthermore, by Prop. 3.2.2 we have that exists a polynomial $P_{j,k}$ in $j+k$ indeterminates, of degree $j+1$, with rational coefficients such that

$$E_{j,k}(z, L) = \frac{(A(L)/2)^j}{(k-1)(k-2)\cdots(k-j)} P_{j,k}(E_1(z,L), \ldots, E_{j+k}(z,L)).$$

Combining the two relations we obtain that the left-hand side of equation (3.10) is given by $(A(L)/2)^j$ times a linear combination with rational coefficient of $E_k(\psi(\mathfrak{b})\rho, L)$ for $\mathfrak{b} \in B$. Since $E_k(\psi(\mathfrak{b})\rho, L)$ is algebraic over K we deduce that the right-hand side divided by $(A(L)/2)^j$ is algebraic. Recall that the area of L is given by

$$A(L) = |\Omega_\infty|^2 A(\mathcal{O}_K) = \frac{|\Omega_\infty|^2}{2\pi i}\sqrt{d_K}.$$

We then conclude that

$$\left(\frac{2\pi}{\sqrt{d_K}}\right)^j \Omega_\infty^{-(k+j)} L_{K(\mathfrak{f})}(\overline{\psi}^{k+j}, k)$$

is algebraic over K. □

Definition 3.13. *Let I denote the set of integral ideals of K which are prime to $6\mathfrak{p}\mathfrak{f}$, and let*

$$\mathcal{S} = \left\{ \mu : I \to \mathbb{Z} \,|\, \mu(\mathfrak{a}) = 0 \text{ for almost all } \mathfrak{a} \in I \text{ and } \sum_{\mathfrak{a}\in I}(N\mathfrak{a} - 1)\mu(\mathfrak{a}) = 0 \right\}.$$

If $\mu \in \mathcal{S}$, we set

$$\Theta(z;\mu) = \prod_{\mathfrak{a}\in I} \Theta(z,\mathfrak{a})^{\mu(\mathfrak{a})}, \quad \Lambda_m(z;\mu) = \prod_{\mathfrak{a}\in I} \Lambda_m(z,\mathfrak{a})^{\mu(\mathfrak{a})}.$$

Lemma 3.3.2. *Let $\mu \in \mathcal{S}$. Then for each integers $n, m \geq 0$, we have*

$$\prod_{\eta\in\mathfrak{p}^{-n}L/L} \Lambda_m(z+\eta;\mu) = \Lambda_m(\pi^n z;\mu)$$

where the product is taken over a set $\{\eta\}$ of representatives modulo L of the π^n-division points of L.

Proof. Since π^n is the generator of the ideal $\mathfrak{p}^n$, it follows from Distribution relation Theorem 3.2 that

$$\prod_{\eta\in\mathfrak{p}^{-n}L/L} \Lambda_m(z+\eta;\mu) = \prod_{\eta\in\mathfrak{p}^{-n}L/L}\prod_{\mathfrak{b}\in B_m} \Theta(z+\eta+\psi(\mathfrak{b})\rho_m;\mu) =$$
$$= \prod_{\mathfrak{b}\in B_m} \Theta(\pi^n z + \pi^n\eta + \psi(\mathfrak{b}\mathfrak{p}^n)\rho_m;\mu).$$

Observe that $\mathfrak{b}\mathfrak{p}^n$ defines a set of representatives $B_{m'}$ for the Galois group $Gal(H_m/F_m)$. By Lemma 3.3.1 we conclude

$$\prod_{\eta\in\mathfrak{p}^{-n}L/L} \Lambda_m(z+\eta;\mu) = \Lambda_m(\pi^n z;\mu).$$

□

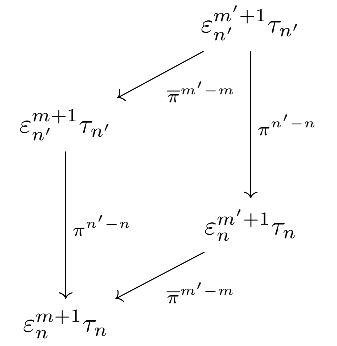
Let T_π denote the Tate module $\varprojlim \widehat{E}_{\pi^{n+1}}$, where the limit is taken relative to the usual projection maps given by multiplication by powers of π. We fix a generator $u = (u_n)$ as $\mathcal{O}_\mathfrak{p}$ module of T_π, i.e $[\pi](u_{n+1}) = u_n$. We can for example fix u_n to be $\varepsilon(\Omega_\infty/\pi^n) = -2\wp(\Omega_\infty/\pi^n)/\wp'(\Omega_\infty/\pi^n)$. Let $\tau_n \in \mathbb{C}$ such that $u_n = \varepsilon(\tau_n)$, where $\varepsilon(z)$ is given by $\varepsilon(z) = -2\wp(z)/\wp'(z)$ as discussed in Section 2.2.1. Since $[\pi^{n+1}]u_n = 0$, we have that $\pi^{n+1}\tau_n \in L$. Since $\overline{\pi}$ is a unit in $\varprojlim \mathcal{O}_K/\mathfrak{p}^n \cong \mathbb{Z}_p$ we can choose $\varepsilon_n \in \mathcal{O}_K$ such that $\varepsilon_n\overline{\pi} \equiv 1 \mod \mathfrak{p}^{n+1}$ and obtaining

$$\varepsilon(\varepsilon_n^{m+1}\tau_n) = [\varepsilon_n^{m+1}]u_n = [\overline{\pi}^{-(m+1)}]u_n$$

We can then construct $\mathfrak{f}\overline{\mathfrak{p}}^{m+1}\mathfrak{p}^{n+1}$-division points taking $\varepsilon_n^{m+1}\tau_n + \rho_m$, indeed we have

$$[f\overline{\pi}^{m+1}\pi^{n+1}]\varepsilon(\varepsilon_n^{m+1}\tau_n + \rho_m) = [f\overline{\pi}^{m+1}\pi^{n+1}][\overline{\pi}^{-(m+1)}]u_n[+][f\overline{\pi}^{m+1}\pi^{n+1}]\varepsilon(\rho_m) = 0.$$

By construction, we have the following diagram.



Definition 3.14. *We call elliptic units $C'_{n,m}$ the subgroup of the units of $K_{n,m}$ generated by $\Theta(\varepsilon_n^{m+1}\tau_n+\rho_m;\mu)$ for all $\mu\in\mathcal{S}$.*

Lemma 3.3.3. *$C'_{n,m}$ is stable under the action of the Galois group $Gal(H_{n,m}/K)$. In particular, $C'_{n,m}$ is independent of the choice of the primitive $\mathfrak{f}\mathfrak{p}^{n+1}\overline{\mathfrak{p}}^{m+}$-division point $\rho_{n,m}$ of L.*

Proof. Let $\mathfrak{b}$ be an arbitrary integral ideal of K, prime to S and p, consider $\sigma_{\mathfrak{b}}=(\mathfrak{b},H_{n,m}/K)$ the Artin symbol of $\mathfrak{b}$. Let β any generator of $\mathfrak{b}$. Thus by Theorem 3.1 we have

$$\Theta(\rho_{n,m};\mu)^{\sigma_{\mathfrak{b}}}=\Theta(\beta\rho_{n,m};\mu).$$

Since $\mathfrak{b}$ is coprime with $\mathfrak{f}$, $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ we deduce the values of Θ at two different primitive $\mathfrak{f}\mathfrak{p}^{n+1}\overline{\mathfrak{p}}^{m+}$-division points belong to the same orbit under the action of $Gal(H_{n,m}/K)$. By Corollary 3.4.1 we obtain

$$\begin{aligned}\Theta(\rho_{n,m};\mu)^{\sigma_{\mathfrak{b}}}&=\prod_{\mathfrak{a}\in I}(\Theta(\rho_{n,m},\mathfrak{a})^{\sigma_{\mathfrak{b}}})^{\mu(\mathfrak{a})}=\prod_{\mathfrak{a}\in I}(\Theta(\rho_{n,m},\mathfrak{a}\mathfrak{b})\Theta(\rho_{n,m},\mathfrak{b})^{-N\mathfrak{a}})^{\mu(\mathfrak{a})}=\\&=\Theta(\rho_{n,m},\mathfrak{b})^{-N\mathfrak{a}\sum_{\mathfrak{a}\in I}\mu_{\mathfrak{a}}}\prod_{\mathfrak{a}\in I}\Theta(\rho_{n,m},\mathfrak{a}\mathfrak{b})^{\mu(\mathfrak{a})}.\end{aligned}$$

Define $\mu':I\to\mathbb{Z}$ as follows

$$\mu'(\mathfrak{a})=\begin{cases}0 & \text{if } \mathfrak{b}\nmid\mathfrak{a}\\ -N\mathfrak{a}\sum_{\mathfrak{a}\in I}\mu(\mathfrak{a}) & \text{if } \mathfrak{a}=\mathfrak{b}\\ \mu(\mathfrak{a}\mathfrak{b}^{-1}) & \text{if } \mathfrak{b}\mid\mathfrak{b},\ \mathfrak{a}\neq\mathfrak{b}.\end{cases}$$

Then we observe

$$\sum_{\mathfrak{a}\in I}(N\mathfrak{a}-1)\mu'(\mathfrak{a})=-N\mathfrak{a}\sum_{\mathfrak{a}\in I}\mu(\mathfrak{a})+\sum_{\mathfrak{a}\mathfrak{b}\in I}(N\mathfrak{a}\mathfrak{b}-1)\mu(\mathfrak{a})=\sum_{\mathfrak{a}\in I}(N\mathfrak{a}-1)\mu(\mathfrak{a})=0$$

and so we conclude $\mu'\in\mathcal{S}$. In particular, we deduce $\Theta(\rho_{n,m};\mu)^{\sigma_{\mathfrak{b}}}=\Theta(\rho_{n,m};\mu')$ and then the elliptic units are stable under the Galois action. □

Lemma 3.3.4. *Let $m'\geq m\geq 0$ and $n'\geq n\geq 0$. Then, for each $\mu\in\mathcal{S}$,*

$$N_{H_{n',m'}/H_{n,m}}\Theta(\varepsilon_{n'}^{m'+1}\tau_{n'}+\rho_{m'};\mu)=\Theta^{(\mathfrak{p}^{n'-n},H_m/K)}(\varepsilon_n^{m+1}\tau_n+\rho_m;\mu).$$

Proof. Let $\mathfrak{c}$ be an integral ideal of K, prime to $6p\mathfrak{f}$ whose Artin symbol $\sigma_{\mathfrak{c}}=(\mathfrak{c},H_{n',m'}/K)$ fixes the subfield $H_{n,m}$. Let ρ a $\mathfrak{f}\overline{\mathfrak{p}}^{m+1}\mathfrak{p}^{n+1}$-torsion point of L, then by Corollary B.3.1 we have

$$\xi(\rho)=\xi(\rho)^{\sigma_{\mathfrak{c}}}=\xi(\psi(\mathfrak{c})\rho).$$

We deduce $\rho=\psi(\mathfrak{c})\rho+\gamma$ with $\gamma\in L=\Omega_\infty\mathcal{O}_K$. Since $\rho\in L/f\pi^{n+1}\overline{\pi}^{m+1}$ we obtain

$$\psi(\mathfrak{c})\equiv 1 \mod \mathfrak{f}\mathfrak{p}^{n+1}\overline{\mathfrak{p}}^{m+1}.$$

Considering now the primitive $\mathfrak{f}\mathfrak{p}^{n'+1}\overline{\mathfrak{p}}^{m'+1}$-division point $\varepsilon_{n'}^{m'+1}\tau_{n'}+\rho_{m'}$, we can write

$$\begin{aligned}\Theta(\varepsilon_{n'}^{m'+1}\tau_{n'}+\rho_{m'};\mu)^{\sigma_{\mathfrak{c}}}&=\Theta(\psi(\mathfrak{c})\varepsilon_{n'}^{m'+1}\tau_{n'}+\psi(\mathfrak{c})\rho_{m'};\mu)=\\&=\Theta(\varepsilon_{n'}^{m'+1}\tau_{n'}+\rho_{m'}+\delta_{\mathfrak{c}};\mu)\end{aligned}$$

with $\delta_{\mathfrak{c}} = (\psi(\mathfrak{c}) - 1)(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'})$ We deduce $\delta_{\mathfrak{c}}$ is a $\mathfrak{p}^{n'-n}\overline{\mathfrak{p}}^{m'-m}$-division point of L. Hence, every conjugate of $\Theta(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'}; \mu)$ under $Gal(H_{n',m'}/H_{n,m})$ is given by $\Theta(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'} + \delta; \mu)$ for some $\mathfrak{p}^{n'-n}\overline{\mathfrak{p}}^{m'-m}$-division point δ. There are exactly $p^{n'+m'-(n+m)}$ such division points, which is equal to the cardinality of $Gal(H_{n',m'}/H_{n,m})$. We conclude

$$N_{H_{n',m'}/H_{n,m}}\Theta(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'}; \mu) = \prod_{\delta} \Theta(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'} + \delta; \mu)$$

where the product is taken over any set of representatives of $\mathfrak{p}^{n'-n}\overline{\mathfrak{p}}^{m'-m}L$ modulo L. By the Distribution relation Theorem 3.2 we deduce

$$\prod_{\delta} \Theta(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'} + \delta; \mu) = \Theta(\psi(\mathfrak{p}^{n'-n}\overline{\mathfrak{p}}^{m'-m})(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'}); \mu).$$

In particular, since $\psi(\mathfrak{p}^{n'-n}\overline{\mathfrak{p}}^{m'-m}) = \pi^{n'-n}\overline{\pi}^{m'-m}$ we have

$$\psi(\mathfrak{p}^{n'-n}\overline{\mathfrak{p}}^{m'-m})(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'}) = \varepsilon_{n}^{m+1}\tau_{n} + \pi^{n'-n}\rho_{m}.$$

In the previous lemma we proved $\Theta(z+\psi(\mathfrak{p})^{n'-n}\rho_m, \mathfrak{a}) = \Theta^{(\mathfrak{p}^{n'-n}, H_m, K)}(z+\rho_m, \mathfrak{a})$ that implies

$$\Theta(\varepsilon_{n}^{m+1}\tau_{n} + \pi^{n'-n}\rho_{m}; \mu) = \Theta^{(\mathfrak{p}^{n'-n}, H_m, K)}(z + \rho_m; \mu)$$

which we conclude

$$N_{H_{n',m'}/H_{n,m}}\Theta(\varepsilon_{n'}^{m'+1}\tau_{n'} + \rho_{m'}; \mu) = \Theta^{(\mathfrak{p}^{n'-n}, H_m/K)}(\varepsilon_{n}^{m+1}\tau_{n} + \rho_m; \mu).$$

□

From Lemma 3.3.4 we deduce the following fundamental corollary.

Corollary 3.7.3. *Let $\mu \in \mathcal{S}$ and consider*

$$e_{n,m}(\mu) = \Lambda_m^{\varphi^{-n}}(z; \mu)|_{z=\varepsilon_n^{m+1}\tau_n}.$$

Then $(e_{n,m}(\mu)) \in U'_{\infty}$.

Proof. First of all observe that by Theorem 3.3 we have $\Lambda_m(\varepsilon_n^{m+1}\tau_n; \mu)$ is a unit in $K_{n,m}$ and so $e_{n,m}(\mu)$ can be regarded as belonging to $U'_{n,m}$. We need to check the norm compatibility. Since $(\mathfrak{p}, H_{m'}/K)$ and φ coincide on $F_{m'}$, by the previous lemma we have

$$\begin{aligned} N_{K_{n',m'}/K_{n,m}}\Lambda_{m'}^{\varphi^{-n'}}(z; \mu)|_{z=\varepsilon_{n'}^{m'+1}\tau'_n} &= \Lambda_m^{(\mathfrak{p}^{n'-n}, H_m/K)\varphi^{-n'}}(z; \mu)|_{z=\varepsilon_n^{m+1}\tau_n} = \\ &= \Lambda_m^{\varphi^{-n}}(z; \mu)|_{z=\varepsilon_n^{m+1}\tau_n}. \end{aligned}$$

Thus, the $e_{n,m}(\mu)$ are compatible with respect to the norm map, and hence $(e_{n,m}(\mu)) \in U'_{\infty}$. □

We write $e(\mu)$ for $(e_{n,m}(\mu))$ and C'_{∞} for the projective limit of $C'_{n,m}$ with respect to the norm maps. We then deduce $e(\mu) \in C'_{\infty}$ for all $\mu \in \mathcal{S}$.

Recall that by Lemma 3.3.1 we have $\Lambda_m(z; \mu)$ is a rational function of $\wp(z)$ and $\wp'(z)$. In particular, $\Lambda_m(z; \mu)$ has a power series expansion with coefficients in F_m and hence in Φ_m.

Theorem 3.8. *In terms of the parameter* $t = -2\wp(z)/\wp'(z)$ *of* $\hat{E}$, $\Lambda_m(z,\mathfrak{a})$ *has an expansion*

$$\Lambda_m(z,\mathfrak{a}) = \sum_{k=0}^{\infty} h_{k,m}(\mathfrak{a})t^k$$

where $h_{k,m}(\mathfrak{a})$ *belong to* $\mathcal{R}_m$, *and* $h_{0,m}(\mathfrak{a})$ *is a unit in* $\mathcal{R}_m$.

Proof. First of all, observe that we have

$$\Lambda_m(0,\mathfrak{a}) = \prod_{\mathfrak{b}\in B_m} \Theta(\psi(\mathfrak{b})\rho_m,\mathfrak{a}) = \prod_{\mathfrak{b}\in B_m} \Theta(\rho_m,\mathfrak{a})^{\sigma_\mathfrak{b}} = N_{H_m/K}(\Theta(\rho_m,\mathfrak{a}))$$

and then by Theorem 3.3 we deduce $h_{0,m}(\mathfrak{a}) = \Lambda_m(0,\mathfrak{a}) \in \mathcal{R}_m^\times$. In order to prove the Lemma we can now show that $\Lambda_m(z,\mathfrak{a})^{-1}$ has a power series expansion in t with coefficients in $\mathcal{R}_m$. Take $\mathfrak{b} \in B_m$ and put $\eta = \psi(\mathfrak{b})\rho_m$. By definition we have

$$\Theta(z+\eta,\mathfrak{a})^{-1} = \frac{\Delta(\mathfrak{a}^{-1}L)}{\Delta(L)^{N\mathfrak{a}}} \prod_{v\in\mathfrak{a}^{-1}L/L-\{0\}} (\wp(z+\eta)-\wp(v))^6$$

where v runs through a set of representatives of the non-zero cosets of $\mathfrak{a}^{-1}L$ modulo L. Let M denote the finite extension of H_m which is obtained by adjoining to H_m all the $\wp(v)$, and let $\mathfrak{P}$ be any prime of M lying above $\mathfrak{p}$. Let $\mathcal{O}_\mathfrak{P}$ be the ring of integers of the completion of M at $\mathfrak{P}$, then we claim that $\Theta(z+\eta)^{-1}$ can be expanded as a power series in t with coefficients in $\mathcal{O}_\mathfrak{P}$. Since E has good reduction at $\mathfrak{p}$ then $\Delta(L)$ is a unit at $\mathfrak{P}$. Furthermore, if α is a generator of $\mathfrak{a}$ then $\Delta(\mathfrak{a}^{-1}L) = \alpha^{12}\Delta(L)$ implies $\Delta(\mathfrak{a}^{-1}L)$ is integral at $\mathfrak{P}$. By the addition theorem we have

$$\wp(z+\eta)-\wp(v) = \frac{1}{4}\left(\frac{\wp'(z)-\wp'(\eta)}{\wp(z)-\wp(\eta)}\right)^2 - \wp(z)-\wp(\eta)-\wp(v).$$

By Lemma 2.2.2 we have $\wp(\eta)$ and $\wp(v)$, and consequently $\wp'(\eta)$ and $\wp'(v)$, lie in $\mathcal{O}_\mathfrak{P}$ since their orders are coprime with $\mathfrak{p}$. In particular recall that there exist two power series $a, b \in 1+t\mathcal{O}_\mathfrak{p}[[t]]$ such that $x = t^{-2}a(t)$, $y = -2t^{-3}a(t)$. Then substituting this expansion into the previous expression we get

$$\wp(z+\eta)-\wp(v) = \frac{1}{4}\left(\frac{-2t^{-3}a(t)-\wp'(\eta)}{t^{-2}a(t)-\wp(\eta)}\right)^2 - t^{-2}a(t)-\wp(\eta)-\wp(v)$$

and since $a(t) = 1+t^2a'(t)$ the terms of negative exponents cancel out and we can conclude

$$\begin{aligned}\wp(z+\eta)-\wp(v) &= \frac{1}{4}\left(\frac{-2-t^2(t\wp'(\eta)+a'(t))}{t(1-t^2(\wp(\eta)-a'(t))}\right)^2 - t^{-2}a(t)-\wp(\eta)-\wp(v) = \\ &= t^{-2}(1+t^2a''(t))(1+t^2a'''(t)) - t^{-2}(1+t^2a'(t)) - \wp(\eta)-\wp(v) = \\ &= -\wp(\eta)-\wp(v)+a'(t)+tc(t)\end{aligned}$$

for $a'(t), a''(t), a'''(t), c(t) \in \mathcal{O}_\mathfrak{P}[[t]]$. Then $\Theta(z+\eta,\mathfrak{a})^{-1}$ has a power series expansion with coefficients in $\mathcal{O}_\mathfrak{P}$. The same is clearly true for $\Lambda_m(z,\mathfrak{a})$. $\square$

3.4 Table of values

In this final section, we provide some numerical values of the Eisenstein numbers in particular cases. We are interested in studying the numerical properties of these values for the lattices attached to elliptic curves with complex multiplication given by $\mathcal{O}_K$.

$L = \Omega_\infty(\mathbb{Z} + i\,\mathbb{Z})$ with $\Omega_\infty \approx 1.854074677301371918433850347201 - 1.854074677301371918433850347201 i$

$Q(i)$	0	$\frac{\omega_1}{2}$	$\frac{\omega_1}{3}$	$\frac{\omega_1}{4}$	$\frac{\omega_1}{5}$
E_4	$\frac{1}{15}$	$\frac{2}{3}$	$\frac{2+2\sqrt{3}}{3}$	$\frac{8}{3} + 2\sqrt{2}$	$\frac{8}{3} + 2\sqrt{5} + 2\sqrt{5+2\sqrt{5}}$
E_6	0	$-\frac{2}{5}i$	$-\frac{2i\sqrt{144+86\sqrt{3}}}{15}$	$-\frac{2}{5}i\sqrt{498+352\sqrt{2}}$	$\frac{2}{5}i\sqrt{2(1779+821\sqrt{5}+\sqrt{6527405+2917702\sqrt{5}})}$
E_8	$-\frac{1}{525}$	$-\frac{8}{35}i$	$-\frac{164}{105} - \frac{4\sqrt{3}}{5}$	$-\frac{512}{35} - \frac{52\sqrt{2}}{5}$	$-\frac{4}{35}(408+161\sqrt{5}+7\sqrt{5945+2638\sqrt{5}})$
E_{10}	0	$-\frac{2}{15}$	$\frac{2}{15}i\sqrt{416+\frac{722}{\sqrt{3}}}$	$\frac{2}{15}i\sqrt{131058+92672\sqrt{2}}$	$\frac{2}{15}i\sqrt{2(2838999+1272121\sqrt{5}+\sqrt{16151310740405+7223071972862\sqrt{5}})}$

$L = \Omega_\infty(\mathbb{Z} + \sqrt{-2}\,\mathbb{Z})$ with $\Omega_\infty = -0.17382248014992879654865318312 2i$

$Q(\sqrt{-2})$	0	$\frac{\omega_1}{2}$	$\frac{\omega_1}{3}$	$\frac{\omega_1}{4}$
E_4	-2450	-7350	$14700(\sqrt{6}-4)$	$-29400(1+\sqrt{2})$
E_6	68600	-617400	$617400(19-10\sqrt{6})$	$-1234800(8+5\sqrt{2})$
E_8	-2572500	-38587500	$-30870000(49\sqrt{6}+136)$	$-246960000(10+7\sqrt{2})$
E_{10}	$\frac{840350000}{11}$	-2521050000	$2521050000(451-196\sqrt{6})$	$-20168400000(32+23\sqrt{2})$

$L = \Omega_\infty(\mathbb{Z} + \frac{1+\sqrt{-3}}{2}\,\mathbb{Z})$ with $\Omega_\infty \approx 2.1032731579881813917625286185 8 - 1.2143253239437908059099708448 9i$

$Q(\sqrt{-3})$	0	$\frac{\omega_1}{2}$	$\frac{\omega_1}{3}$	$\frac{\omega_1}{4}$
E_4	0	-1	$-2\sqrt[3]{2}$	$-2(2+\sqrt{3})$
E_6	$\frac{1}{35}$	$\frac{3}{5}$	$\frac{18}{5}$	$\frac{48}{5} + 6\sqrt{3}$
E_8	0	$-\frac{3}{7}$	$-\frac{24\sqrt[3]{4}}{7}$	$-\frac{192}{7} - \frac{108\sqrt{3}}{7}$
E_{10}	0	$\frac{2}{7}$	$\frac{46\sqrt[3]{2}}{7}$	$\frac{512}{7} + \frac{298\sqrt{3}}{7}$

$L = \Omega_\infty(\mathbb{Z} + \frac{1+\sqrt{-7}}{2}\,\mathbb{Z})$ with $\Omega_\infty \approx 0.24958946796272557567257864184 6$

$Q(\sqrt{-7})$	0	$\frac{\omega_1}{2}$	$\frac{\omega_1}{3}$	$\frac{\omega_1}{4}$
E_4	-525	$\frac{525}{2} - \frac{1575\sqrt{-7}}{2}$	$-5250+1050\sqrt{21}-3150\sqrt{2(3-\sqrt{21})}$	$-2100-6300\sqrt{-7}$
E_6	-9450	$-33075+23625\sqrt{-7}$	$-859950+141750\sqrt{21}+$ $15750\sqrt{14(-79\sqrt{21}-333)}$	$-1190700+661500\sqrt{-7}$
E_8	-118125	$\frac{3898125}{2} - \frac{354375\sqrt{-7}}{2}$	$-96862500+26932500\sqrt{21}-$ $1417500\sqrt{-42(61\sqrt{21}-279)}$	$263655000-$ $25515000\sqrt{-7}$
E_{10}	$-\frac{24806250}{11}$	$-62015625-$ $12403125\sqrt{-7}$	$-16644993750+3249618750\sqrt{21}-$ $8268750\sqrt{-1581522\sqrt{21}-7235766}$	$-31107037500-$ $6003112500\sqrt{-7}$

$L = \Omega_\infty(\mathbb{Z} + \frac{1+\sqrt{-11}}{2}\,\mathbb{Z})$ with $\Omega_\infty \approx 0.15798806243604140684722654209 1$

$Q(\sqrt{-11})$	0	$\frac{\omega_1}{2}$	$\frac{\omega_1}{3}$
E_4	$-\frac{17248}{5}$	$\sim 199.1-4551.7i$ $(x^3+51744x^2+1082355740928=0)$	$-4312+4312\sqrt{-11}$
E_6	$-\frac{664048}{5}$	$\sim -62921+357641i$ $(125x^3+1045875600x^2+146118025025280x+1358411351947819868 16=0)$	$-\frac{1328096}{5} + \frac{2656192\sqrt{-11}}{5}$
E_8	$-\frac{127497216}{25}$	$\sim 9450795-12681946i$ $125x^3+162558950400x^2-3086012688533913600x+41255030382161114985725952=0$	$\frac{733108992}{5} - \frac{223120128\sqrt{-11}}{5}$
E_{10}	$\frac{1041227264}{5}$	$\sim -820896839+123463390i$ $125x^3+26629387276800x^2+4346916326233023578112 0x+182092826494966272705986423685 12=0$	$-\frac{123125123968}{5} + \frac{8069511296\sqrt{-11}}{5}$

In the proof of Lemma 3.8 we have proven that the expansion of $\Theta(\eta + z, \mathfrak{a})$ is $\mathfrak{p}$-integral for $\mathfrak{a}$ integral ideal prime to $6\mathfrak{f}$ and η a primitive $\mathfrak{m}$-torsion point. Recall that by Theorem 3.6, $\Theta(z, \mathfrak{a})$ has a Lauren expansion given by

$$\log \Theta(z+\eta, \mathfrak{a}) = 12 \sum_{k \geq 0} (-1)^k (k-1)! (N\mathfrak{a} E_k(\eta, L) - E_k(\eta, \mathfrak{a}^{-1}L)) z^k.$$

The following tables show the integrality of the coefficients.

$L = \Omega_\infty(\mathbb{Z} + i\,\mathbb{Z})$ with $\Omega_\infty \approx 1.854074677301371918433850347201 - 1.854074677301371918433850347201 i$. We consider $\mathfrak{a} = 7\mathcal{O}_K$ and $\mathfrak{m} = 2\mathcal{O}_k$ with torsion point $\eta = w_1/2$.

$Q(i)$	$E_k(w_1/2, L)$	$E_k(w_1/2, 7^{-1}L)$	$(k-1)!(49 \cdot E_k(w_1/2, L) - E_k(w_1/2, 7^{-1}L))$
E_4	$\frac{2}{3}$	$\frac{4802}{3}$	-9408
E_6	$-\frac{2}{5}i$	$-\frac{235298}{5}i$	$5644800i$
E_8	$-\frac{8}{35}$	$-\frac{6588344}{5}$	6640994304
E_{10}	$\frac{2}{15}i$	$\frac{564950498}{15}i$	$-13667280076800i$

$L = \Omega_\infty(\mathbb{Z} + \frac{1+\sqrt{-3}}{2}\mathbb{Z})$ with $\Omega_\infty \approx 2.10327315798818139176252861858 - 1.21432532394379080590997084489i$. We consider $\mathfrak{a} = 7\mathcal{O}_K$ and $\mathfrak{m} = 2\mathcal{O}_k$ with torsion point $\eta = w_1/2$.

$Q(\sqrt{-3})$	$E_k(w_1/2, L)$	$E_k(w_1/2, 7^{-1}L)$	$(k-1)!(49 \cdot E_k(w_1/2, L) - E_k(w_1/2, 7^{-1}L))$
E_4	-1	-2401	14112
E_6	$\frac{3}{5}$	$\frac{352947}{5}$	-8467200
E_8	$-\frac{3}{7}$	-2470629	12451859280
E_{10}	$\frac{2}{7}$	80707214	-29287028736000

We consider $\mathfrak{a} = 7\mathcal{O}_K$ and $\mathfrak{m} = 3\mathcal{O}_k$ with torsion point $\eta = w_1/3$.

$Q(\sqrt{-3})$	$E_k(w_1/3, L)$	$E_k(w_1/3, 7^{-1}L)$	$(k-1)!(49 \cdot E_k(w_1/3, L) - E_k(w_1/3, 7^{-1}L))$
E_4	$-2\sqrt[3]{2}$	$-4802\sqrt[3]{2}$	$28224\sqrt[3]{2}$
E_6	$\frac{18}{5}$	$\frac{2117682}{5}$	-50803200
E_8	$-\frac{24\sqrt[3]{4}}{7}$	$-19765032\sqrt[3]{4}$	$99614914560\sqrt[3]{4}$
E_{10}	$\frac{46\sqrt[3]{2}}{7}$	$1856265922\sqrt[3]{2}$	$-67360144320000\sqrt[3]{2}$

$L = \Omega_\infty(\mathbb{Z} + \frac{1+\sqrt{-7}}{2}\mathbb{Z})$ with $\Omega_\infty \approx 0.24958946796272557567257864184$6. We consider $\mathfrak{a} = 11\mathcal{O}_K$ and $\mathfrak{m} = 2\mathcal{O}_k$ with torsion point $\eta = w_1/2$.

$Q(\sqrt{-7})$	$E_k(w_1/2, L)$	$E_k(w_1/2, 11^{-1}L)$	$(k-1)!(121 \cdot E_k(w_1/2, L) - E_k(w_1/2, 11^{-1}L))$
E_4	$\frac{525}{2} - \frac{1575\sqrt{-7}}{2}$	$\frac{7686525}{2} + \frac{23059575\sqrt{-7}}{2}$	$-22869000 - 68607000\sqrt{-7}$
E_6	$-33075 + 23625\sqrt{-7}$	$-58594380075 - 41853128625\sqrt{-7}$	$7030845360000 + 5022032400000\sqrt{-7}$
E_8	$\frac{3898125}{2} - \frac{354375\sqrt{-7}}{2}$	$\frac{835597712998125}{2} + \frac{75963428454375\sqrt{-7}}{2}$	$-2105705048139000000 - 191427731649000000\sqrt{-7}$
E_{10}	$-62015625 - 12403125\sqrt{-7}$	$-1608525597521390625 + 321705119504278125\sqrt{-7}$	$583701766105550400000000 + 116740353221110080000000\sqrt{-7}$

By Corollary 3.6.1 we know that the Eisenstein numbers at $\mathfrak{m}$-torsion point are algebraic. Since they live in the field $K(E_\mathfrak{m})$, the degree of the extension becomes quickly extremely higher. In particular, using a Computer Algebra System like GP/Pari, naive methods to detect the algebraicity of these values fail even for 5 torsion points. In order to avoid this problem, we can use the property of the action of the Galois action and detect the algebraicity through the use of symmetric polynomials. Recall in particular that by Corollary 3.6.1 we have

$$E_k(\rho, L)^{\sigma_\mathfrak{b}} = E_k(\psi(\mathfrak{b})\rho, L)$$

for $\mathfrak{b}$ coprime with $\mathfrak{m}$. In particular, the Galois elements fix the Newton sums

$$p_r(E_k) = \sum_{\substack{\rho \in \mathfrak{m}^{-1}L/L \\ \rho \text{ primitive}}} E_k(\rho, L)^r.$$

The following tables show the rationality of these values for some examples. The code to compute the Newton sums can be found in Appendix C

$L = \Omega_\infty(\mathbb{Z} + i\,\mathbb{Z})$ with $\Omega_\infty \approx 1.85407467730137191843385034720 - 1.85407467730137191843385034720i$. We consider $\mathfrak{m} = 6\mathcal{O}_k$.

$Q(i)$	p_1	p_2	p_3	p_4
E_4	$\frac{259}{3}$	$\frac{28937}{9}$	$\frac{2209807}{27}$	$\frac{185649185}{81}$
E_6	0	$-\frac{2029832}{25}$	0	$\frac{5737870519264}{3375}$
E_8	$-\frac{47989}{15}$	$\frac{510611417}{225}$	$-\frac{281097320777293}{165375}$	$\frac{887319825715670783}{694575}$
E_{10}	0	$\frac{13931542792}{225}$	0	$\frac{29170111094091269344}{30375}$

$L = \Omega_\infty(\mathbb{Z}+\sqrt{-2}\,\mathbb{Z})$ with $\Omega_\infty \approx -0.17382248014992879654865318312 2i$. We consider $\mathfrak{m} = 6\mathcal{O}_k$.

$Q(\sqrt{-2})$	p_1	p_2	p_3	p_4
E_4	-3172750	4342358562500	-5859740610595375000	$8224911586379185906250000$
E_6	3200533000	5821228519274000000	$9662470329903512206120000000$	$16381256601707174904959423324000000000$
E_8	-4320809587500	$8153357340912439556250000$	$-16376688311616507014921415796875000000$	$32998281437217320249737406291129961914062500000000$
E_{10}	4619340923750000	$1154449132170414574392500000000 0$	$27693386494431987136003527529076375000000000000$	$665032573435538333248185554540521050697251593750000000000000000$

$L = \Omega_\infty(\mathbb{Z}+\frac{1+\sqrt{-3}}{2}\,\mathbb{Z})$ with $\Omega_\infty \approx 2.10327315798818139176252861858-1.21432532394379080590997084489i$. We consider $\mathfrak{m} = 6\mathcal{O}_K$

$Q(\sqrt{-3})$	p_1	p_2	p_3	p_4
E_4	0	0	311811	0
E_6	1333	310739	$\frac{1763551453}{25}$	$\frac{2005171249303}{125}$
E_8	0	0	$-\frac{5499412144209}{343}$	0
E_{10}	0	0	$\frac{1250350064581656}{343}$	0

$L = \Omega_\infty(\mathbb{Z}+\frac{1+\sqrt{-7}}{2}\,\mathbb{Z})$ with $\Omega_\infty \approx 0.249589467962725575672578641846$. We consider $\mathfrak{m} = 6\mathcal{O}_K$

$Q(\sqrt{-7})$	p_1	p_2	p_3	p_4
E_4	679875	199394015625	-74081487057046875	25260535322103744140625
E_6	-440889750	73646077235062500	$-14326041813121575219375000$	$27746428423505564815307429062 50000$
E_8	-198404521875	25050210647264553515625	$-27738213015525972385166784667 96875$	$30948642432614993042488665534294204711914 0625$
E_{10}	-136358095781250	$831816033156982453535156 2500$	$-53553032546485806472786099290747070312 5000$	$345167365171622115825654511637982203069595336914 06250000$

CHAPTER 4

Coleman Theory

4.1 Coleman power series

In the first chapter, we have studied the properties of the tower field K_∞ and F_∞ generated from K adding the $\mathfrak{p}$ and $\overline{\mathfrak{p}}$-torsion point of the elliptic curve associated to K in complete analogy with the cyclotomic tower $\mathbb{Q}(\mu_{p^\infty})$. The theta function has given us a way to construct a sequence of norm-coherent units in the elliptic tower field. We will now study the construction of the canonical interpolation series for norm-compatible systems of elements in these towers. In 1977, Coates and Wiles [CW77] constructed ad hoc a particular series for the quadratic imaginary case and elliptic units. A few years later, Coleman [Col79] found a conceptual proof that is valid for arbitrary Lubin-Tate groups. An extensive treatment of the cyclotomic case can be found in [CS06].

4.1.1 General results

Let K be a fixed local field, and let $\mathcal{O}_K$ be the ring of integers of K. Fix a uniformizer π of $\mathcal{O}_K$ and let F be a Lubin-Tate formal group with endomorphism ring $\mathcal{O}_K$. For $b \in \mathcal{O}_K$ we write $[b]$ for the endomorphism of F given by b, and Λ_n for the kernel of the endomorphism $[\pi^{n+1}]$. Now take H to be a complete, unramified extension of K and let φ be the Frobenius automorphism of H/K. We define the tower of fields

$$H_n = H(\Lambda_n).$$

When $m \geq n$ we write $N_{m,n}$ for the norm map from H_m to H_n. Fix a generator $v = (v_n)$ of the Tate module $\varprojlim \Lambda_n$ as an $\mathcal{O}_K - module$, i.e., v_n generates Λ_n as an $\mathcal{O}_K$ module and $[\pi](v_{n+1}) = v_n$ for each $n \geq 0$. Let $\mathcal{O}_H$ be the ring of integers of H. For brevity, we will write $\mathcal{O}_n$ the ring of integers of H_n and with $\mathfrak{p}_n$ its maximal ideal.

Theorem 4.1. *Let $\alpha = (\alpha_n)$ be an element of $\varprojlim H_n^\times$ where the limit is taken with respect to the norm maps. Then there exists a unique power series $c_\alpha(T)$ in $\mathcal{O}_H((T))$ satisfying*

$$c_\alpha^{\varphi^{-n}}(v_n) = \alpha_n$$

for all $n \geq 0$.

Proof. See [Col79]. □

Corollary 4.1.1. *Let* α, α' *be elements of* $\varprojlim H_n^\times$. *Then the following properties hold*

(i) $c_{\alpha\alpha'} = c_\alpha c_{\alpha'}$,

(ii) $c_\alpha^\varphi([\pi](T)) = \prod_{\lambda\in\Lambda_0} c_\alpha(T[+]\lambda)$,

(iii) $c_\alpha(0)^{1-\varphi^{-1}} = N_{H_0,H}(\alpha_0)$,

(iv) *if* $\sigma \in Gal(H_\infty/H)$ *and* $k(\sigma) \in \mathcal{O}_K^\times$ *defined in Theorem 2.3 then we have*

$$c_{\sigma(\alpha)} = c_\alpha \circ [k(\sigma)]_f.$$

Proof. (i) Observe that for every $n \geq 0$ we have

$$(c_\alpha c_{\alpha'})^{\varphi^{-n}}(v_n) = c_\alpha^{\varphi^{-n}}(v_n) c_{\alpha'}^{\varphi^{-n}}(v_n) = \alpha_n \alpha'_n.$$

By uniqueness of the Coleman power series stated in the previous theorem we have $c_{\alpha\alpha'} = c_\alpha c_{\alpha'}$.

(ii) Observe that for every $n \geq 1$ we have

$$\alpha_{n-1} = N_{n,n-1}\alpha_n = \prod_{\tau\in Gal(H_n/H_{n-1})} c_\alpha(v_n)^\tau = \prod_{\tau\in Gal(H_n/H_{n-1})} c_\alpha(v_n^\tau).$$

Since $Gal(H_i/H_{i-1})$ acts transitively on the elements of $\Lambda_i - \Lambda_{i-1}$, we deduce that the right-hand side of the previous equation coincides with

$$\prod_{\lambda\in\Lambda_0} c_\alpha(v_n[+]\lambda).$$

Let $g(T) = c_\alpha^\varphi([\pi](T))$, then we have for every $n \geq 0$

$$g^{\varphi^{-n}}(v_n) = c_\alpha^{\varphi^{-n+1}}([\pi](v_n)) = c_\alpha^{\varphi^{-n+1}}(v_{n-1}) = \alpha_{n-1}.$$

By the uniqueness of the Coleman power series stated in the previous theorem we have

$$c_\alpha^\varphi([\pi](T)) = \prod_{\lambda\in\Lambda_i} c_\alpha(T[+]\lambda).$$

(iii) Evaluating the equality (ii) in 0 we obtain

$$c_\alpha^\varphi(0) = \prod_{\lambda\in\Lambda_0} c_\alpha(\lambda) = c_\alpha(0)\cdot N_{H_0/H}(c_\alpha(v_0)) = c_\alpha(0)\cdot N_{H_0/H}(\alpha_0).$$

(iv) Recall from Theorem 2.3 we have that for $\tau \in Gal(H_\infty/H)$ there exists $k(\tau) \in \mathcal{O}_K^\times$ such that $\lambda^\tau = [k(\tau)]\lambda$ for every $\lambda \in \Lambda$. In particular, from the definition of the Coleman power series we have for every $n \geq 0$

$$\tau(\alpha_n) = \tau(c_\alpha(v_n)) = c_\alpha(v_n^\tau) = c_\alpha([k(\tau)]v_n).$$

By uniqueness we conclude $c_{\tau(\alpha)} = c_\alpha \circ [k(\tau)]$.

□

Observe that by uniqueness we have that the Coleman power series associated with $1 \in U_\infty$ is $c_1(T) = 1$. In particular, by the previous corollary, point (ii), we deduce

$$1 = c_1 = c_\alpha c_{\alpha^{-1}}$$

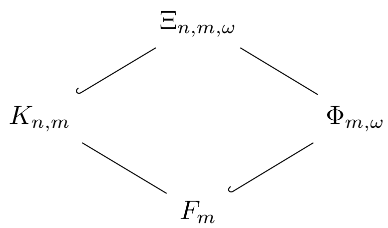
and then c_α is a unit in $\mathcal{O}_H[[T]]$.

4.1.2 Norm coherent units in U_∞

In Section 2.2 we have studied the structure of the extensions of K with elliptic torsion points, recall that we have

$$F_m = K(E_{\overline{\pi}^{m+1}}), \quad K_{n,m} = F_m(E_{\pi^n})$$

and we have fixed the completions $\Xi_{n,m,\omega}$ of $K_{n,m}$ and $\Phi_{m,\omega}$ of F_m at the prime ω above $\mathfrak{p}$. We denoted $\mathcal{R}_{m,\omega}$ the ring of integers of $\Phi_{m,\omega}$.



We have denoted by $U'_{n,m,\omega}$ the units of $\Xi_{n,m,\omega}$ and $U'_{n,m} = \prod_\omega U'_{n,m,\omega}$ where the product is taken over the set of primes ω of F_m lying above $\mathfrak{p}$. The multiplicative groups of units are given by

$$U'_\infty = \varprojlim U'_{n,m}, \quad U_\infty = \varprojlim U_{n,m}$$

where the projective limit is taken with respect to the norm maps on the $\Xi_{n,m}$. As before we denote by ϕ the Frobenius automorphism of the unramified extension F_m/K. Let T_π denote the Tate module $\varprojlim \widehat{E}_{\pi^{n+1}}$, where the limit is taken relative to the usual projection maps given by multiplication by powers of π. We fix a generator $u = (u_n)$ as $\mathcal{O}_\mathfrak{p}$ module of T_π, i.e $[\pi](u_{n+1}) = u_n$. We can for example fix u_n to be $\varepsilon(\Omega_\infty/\pi^n) = -2\wp(\Omega_\infty/\pi^n)/\wp'(\Omega_\infty/\pi^n)$.

By the final remark in Section 2.2.1, the elliptic curve law defines a Lubin-Tate formal group over K, then we can rewrite the previous results as follows.

Theorem 4.2. *Let $\beta = (\beta_{n,m,\omega})$ be an element of U'_∞. Then for each integer $m \geq 0$ and each prime ω of F_m lying above $\mathfrak{p}$, there exists a unique power series $c_{m,\omega,\beta}(T) \in \mathcal{R}_{m,\omega}[[T]]$ satisfying*

$$c^{\varphi^{-n}}_{m,\omega,\beta}(u_n) = \beta_{n,m,\omega}$$

for all $n \geq 0$. Furthermore, we have the following properties

i) $c_{m,\omega,\beta\beta'} = c_{m,\omega,\beta} c_{m,\omega,\beta'}$, *for every* $\beta, \beta' \in U'_\infty$,

ii) $c^{\varphi}_{m,\omega,\beta}([\pi](T)) = \prod_{\eta \in \widehat{E}_\pi} c_{m,\omega,\beta}(T[+]\eta)$

iii) $c_{m,\omega,\beta}(0)^{1-\varphi^{-1}} = N_{K_{1,m},F_m}(\beta_{1,m})$.

We will denote $c_{m,\beta} = (c_{m,\omega,\beta})_\omega \in \prod_\omega \mathcal{R}_{m,\omega}[[T]] = \mathcal{R}_m[[T]]$ the collection of power series for each prime ω of F_m lying above $\mathfrak{p}$.

The previous theorem gives us the existence of the power series c_m for the tower field above F_m. The following result shows the norm relation between the c_m's.

Lemma 4.1.1. *For $m' \geq m$ and ω' a prime of $F_{m'}$ lying above the prime ω of F_m, let $N_{m',m}$ denote the norm map from $\mathcal{R}_{m'}[[T]]$ to $\mathcal{R}_m[[T]]$. Then we have the following identity*

$$c_{m,\beta}(T) = N_{m',m}(c_{m',\beta}(T)).$$

Proof. First of all, denoting $N^n_{m',m}$ the norm map from $\Xi_{n,m',\omega'}$ to $\Xi_{n,m,\omega}$, the structure of U'_∞ implies the identity

$$\prod_{\omega'|\omega} N^n_{m',m}(\beta_{n,m',\omega'}) = \beta_{n,m,\omega}$$

for every $n \geq 0$. In particular, we can write

$$\prod_{\omega'|\omega} N^n_{m',m}(c^{\varphi^{-n}}_{m',\omega',\beta}(v_n)) = \beta_{n,m,\omega}.$$

Since $v_n \in \Xi_{n,m,\omega}$ and $c_{m',\omega',\beta}$ has coefficients in $\mathcal{R}_{m',\omega'}$ we deduce that the norm map acts as follows

$$\prod_{\omega'|\omega} \left(\prod_{\sigma \in Gal(\Phi_{m',\omega'}/\Phi_{m,\omega})} c^{\sigma}_{m',\omega',\beta} \right)^{\varphi^{-n}} (v_n) = \beta_{n,m,\omega}.$$

From the uniqueness of the Coleman power series, we conclude

$$c_{m,\beta}(T) = N_{m',m}(c_{m',\beta}(T)).$$

□

Recall that the logarithm map $\lambda(T)$ associated with $\widehat{E}$ formal group has the property that $\lambda'(T)$ is a unit in the ring $\mathbb{Z}_p[[T]]$. Furthermore, from the remark of Corollary 4.1.1, $c_{m,\omega,\beta}(T)$ is a unit in $\mathcal{R}_{m,\omega}[[T]]$. The following definition introduces the logarithmic derivative of the Coleman power series

Definition 4.1. *We denote by $g_{m,\beta}(T)$ the element of $\mathcal{R}_m[[T]]$ whose ω-component $(g_{m,\beta}(T))_\omega$ is given by*

$$g_{m,\omega,\beta}(T) = \lambda'(T)^{-1} \frac{d}{dT} \log c_{m,\omega,\beta}(T)$$

and it is the logarithmic derivative of the Coleman power series.

Observe that if $\beta = (\beta_{n,m,\omega}) \in U'_\infty$, then we can canonically write using the Teichmuller character

$$\beta_{n,m,\omega} = \omega_{n,m,\omega}(\beta)\langle \beta_{n,m,\omega} \rangle$$

where $\langle \beta_{n,m,\omega} \rangle$ belongs to $U_{n,m,\omega}$ and $\omega_{n,m,\omega}(\beta)$ is a root of unity in $\Xi_{n,m,\omega}$. By the multiplicativity of the Teichmuller character, we have $\langle \beta \rangle = (\langle \beta_{n,m,\omega} \rangle) \in U_\infty$.

Lemma 4.1.2. *For every* $\beta = (\beta_{n,m,\omega}) \in U'_\infty$ *we have*

$$g_{m,\beta}(T) = g_{m,\langle\beta\rangle}(T).$$

Proof. By Proposition 2.2.3, we have that $\Xi_{n,m,\omega}$ is totally ramified over $\Phi_{m,\omega}$. In particular we have $\omega_{n,m,\omega}(\beta) \in \Phi_{m,\omega}$. The Coleman power series for each pair m and ω is then given by the constant series $\omega_{0,m,\omega}(\beta) \in \mathcal{R}_{m,\omega}[[T]]$. By Corollary 4.1.1 we can write

$$c_{m,\omega,\beta}(T) = \omega_{0,m,\omega}(\beta) \cdot c_{m,\omega,\langle\beta\rangle}(T).$$

Since $\omega_{0,m,\omega}$ is a root of unity, we have

$$\log c_{m,\omega,\beta}(T) = \log c_{m,\omega,\langle\beta\rangle}(T)$$

and then applying the definition of $g_{m,\beta}$ we obtain the identity. □

Lemma 4.1.3. *Let* $m' \geq m \geq 0$ *and let* $Tr_{m',m}$ *denote the trace map from* $\mathcal{R}_{m'}[[T]]$ *to* $\mathcal{R}_m[[T]]$*. Then for each* $\beta \in U'_\infty$ *we have*

$$g_{m,\beta}(T) = Tr_{m',m}(g_{m',\beta}(T))$$

and $g_{m,\beta}$ *satisfies the functional equation*

$$\pi g^{\varphi}_{m,\beta}([\pi]T) = \sum_{\eta \in \hat{E}_\pi} g_{m,\beta}(T[+]\eta).$$

Proof. Recall that from Lemma 4.1.1 we have

$$c_{m,\omega,\beta}(T) = \prod_{\omega'|\omega} \prod_{\sigma \in Gal(\Phi_{m',\omega'}/\Phi_{m,\omega})} c^{\sigma}_{m',\omega',\beta}(T).$$

Applying the logarithm map and observing that the Galois action commutes with it we deduce

$$\log c_{m,\omega,\beta}(T) = \sum_{\omega'|\omega} \sum_{\sigma \in Gal(\Phi_{m',\omega'}/\Phi_{m,\omega})} (\log c_{m',\omega',\beta})^{\sigma}(T).$$

The identity follows from the definition of $g_{m,\omega}$.

We have $\lambda(T[+]\eta) = \lambda(T) + \lambda(\eta) = \lambda(T)$ and hence $d/dT\lambda(T[+]\eta) = \lambda'(T)$ for all $\eta \in \hat{E}_\pi$. Thus

$$\begin{aligned} g_{m,\omega,\beta}(T[+]\eta) &= \left(\frac{d}{dT}\lambda(T[+]\eta)\right)^{-1} \cdot \frac{d}{dT} \log c_{m,\omega,\beta}(T[+]\eta) = \\ &= \lambda'(T)^{-1} \frac{d}{dT} \log c_{m,\omega,\beta}(T[+]\eta) \end{aligned}$$

From Theorem 4.2 we have

$$(c^{\varphi}_{m,\omega,\beta} \circ [\pi])(T) = \prod_{\eta \in \hat{E}_\pi} c_{m,\omega,\beta}(T[+]\eta)$$

and then from the previous equality

$$\begin{aligned} \sum_{\eta \in \hat{E}_\pi} g_{m,\omega,\beta}(T[+]\eta) &= \lambda'(T)^{-1} \sum_{\eta \in \hat{E}_\pi} \frac{d}{dT} \log c_{m,\omega,\beta}(T[+]\eta) = \\ &= \lambda'(T)^{-1} \frac{d}{dT} \log(c^{\varphi}_{m,\omega,\beta} \circ [\pi]). \end{aligned}$$

On the other hand, since $\lambda([\pi]T) = \pi\lambda(T)$ we deduce

$$(g^{\varphi}_{m,\omega,\beta} \circ [\pi]) = \left(\frac{d}{dT}\lambda([\pi]T)\right)^{-1} \cdot \frac{d}{dT}\log(c^{\varphi}_{m,\omega,\beta} \circ [\pi]) =$$
$$= \pi^{-1}\lambda'(T)^{-1}\frac{d}{dT}\log(c^{\varphi}_{m,\omega,\beta} \circ [\pi]).$$

Combining the two equations we conclude

$$\pi g^{\varphi}_{m,\beta}([\pi]T) = \sum_{\eta \in \hat{E}_\pi} g_{m,\beta}(T[+]\eta).$$

□

In the following lemma, we see how the Galois action on the system of units acts on the Coleman power series and its logarithmic derivative by formal multiplication.

Lemma 4.1.4. *Let $\beta \in U'_\infty$ and $m \geq 0$, then for every $\sigma \in G_\infty$ and $n \geq 0$ we have*

$$c_{m,\beta^\sigma} = c^{\sigma}_{m,\beta}([k_1(\sigma)](T)),$$
$$g_{m,\beta^\sigma} = k_1(\sigma) g^{\sigma}_{m,\beta}([k_1(\sigma)](T)).$$

Proof. First of all recall that by Theorem 2.6 and Theorem 2.3 we have

$$u_n^\sigma = [k_1(\sigma)](u_n).$$

Then consider the power series $d(T) = c^{\sigma}_{m,\beta,\omega}([k_1(\sigma)](T)) \in \mathcal{R}_m[[T]]$ and we have

$$d^{\varphi^{-n}}(u_n) = c^{\varphi^{-n}\sigma}_{m,\beta,\omega}([k_1(\sigma)](u_n)) = (c^{\varphi^{-n}}_{m,\beta,\omega}(u_n))^\sigma = \beta^\sigma.$$

By uniqueness of the Coleman power series we conclude $c_{m,\beta^\sigma,\omega^\sigma} = d_m$. By definition of $g_{m,\beta}$ we have

$$g_{m,\beta^\sigma,\omega^\sigma}(T) = \lambda'(T)^{-1}\frac{d}{dT}\left(\log c^{\sigma}_{m,\beta,\omega}([k_1(\sigma)](T))\right)$$

while

$$g^{\sigma}_{m,\beta,\omega}([k_1(\sigma)](T)) = \lambda'([k_1(\sigma)](T))\frac{d}{dT}\left(\log c^{\sigma}_{m,\beta,\omega}([k_1(\sigma)](T))\right) =$$
$$= k_1(\sigma)^{-1}\lambda'(T)^{-1}\frac{d}{dT}\left(\log c^{\sigma}_{m,\beta,\omega}([k_1(\sigma)](T))\right).$$

Combining the two equations we conclude $g_{m,\beta^\sigma} = k_1(\sigma) g^{\sigma}_{m,\beta}([k_1(\sigma)](T))$. □

Recall that we wrote $\mathcal{R}_m = \prod_\omega \mathcal{R}_{m,\omega}$ and write $\varprojlim \mathcal{R}_m$ for the projective limit of the rings $\mathcal{R}_m$ relative to the trace maps. We also put $\mathcal{R}_\infty = \bigcup_{m\geq 0} \mathcal{R}_m$ and denote the completion of $\mathcal{R}_\infty$ by $\hat{\mathcal{R}}_\infty$.

Theorem 4.3. *Let $b \in \varprojlim \mathcal{R}_m$. Then there is a unique power series $h_b(T) \in \hat{\mathcal{R}}_\infty[[T]]$ such that*

$$h_b(T) \equiv \sum_{\sigma \in Gal(F_m/K)} (b^\sigma)_{m,\mathfrak{p}_m}(1+T)^{k_2(\sigma)} \mod ((1+T)^{p^{m+1}} - 1) \tag{4.1}$$

for all $m \geq 0$. Where $(b^\sigma)_{m,\mathfrak{p}_m}$ denotes the $\mathfrak{p}_m$-component of the projection onto $\mathcal{R}_m$ of the image of b under the action of any element of G_∞ whose restriction to F_m is σ.

Proof. First of all observe that by Lemma 2.2.6 we have that if $\theta \in Gal(K_\infty/K)$ is trivial on F_m, then $k_2(\theta) \equiv 1 \mod p^{m+1}$, and hence $(1+T)^{k_2(\sigma)}$ is well defined modulo $((1+T)^{p^{m+1}} - 1)$ for all $\sigma \in Gal(F_m/K)$. Let $m' \geq m$, then by the trace compatibility of an element of $\varprojlim \mathcal{R}_m$ we have

$$(b^\sigma)_{m,\mathfrak{p}_m} = \sum_{\substack{\theta \in Gal(F_{m'}/K) \\ \theta|_{F_m} = \sigma}} (b^\theta)_{m',\mathfrak{p}_{m'}}.$$

Consequently we deduce

$$\sum_{\substack{\theta \in Gal(F_{m'}/K) \\ \theta|_{F_m} = \sigma}} (b^\theta)_{m',\mathfrak{p}_{m'}} (1+T)^{k_2(\theta)} \equiv (b^\sigma)_{m,\mathfrak{p}_m} (1+T)^{k_2(\sigma)} \mod ((1+T)^{p^{m+1}} - 1).$$

Calling $h_{m,b}(T)$ the right hand side of (4.1), by completeness of $\hat{\mathcal{R}}_\infty$ we conclude the sequence $h_{m,b}(T)$ converges in $\hat{\mathcal{R}}_\infty[[T]]$. □

Definition 4.2. *Let $b \in \varprojlim \mathcal{R}_m$ and $j \leq 0$, we define*

$$\delta_j(b) = \left[\left((1+T)\frac{d}{dT} \right)^{-j} h_b(T) \right]_{|T=0} \in \hat{\mathcal{R}}_\infty.$$

Lemma 4.1.5. *Let $b \in \varprojlim \mathcal{R}_m$ and $j \leq 0$, then we have*

$$\delta_j(b) \equiv \sum_{\sigma \in Gal(F_m/K)} k_2(\sigma)^{-j} (b^\sigma)_{m,\mathfrak{p}_m} \mod \mathfrak{p}_\infty^{m+1}$$

where $\mathfrak{p}_\infty$ is the maximal ideal of $\hat{\mathcal{R}}_\infty$.

Proof. By (4.1) we have that for every $m \geq 0$ there exists $f(T) \in \hat{\mathcal{R}}_\infty[[T]]$ such that

$$h_b(T) = \sum_{\sigma \in Gal(F_m/K)} (b^\sigma)_{m,\mathfrak{p}_m} (1+T)^{k_2(\sigma)} + f(T) \cdot ((1+T)^{p^{m+1}} - 1).$$

To prove the congruence we proceed by induction on $-j$. Applying the derivation $(1+T)d/dT$ we obtain

$$\left((1+T)\frac{d}{dT} \right) h_b(T) = \sum_{\sigma \in Gal(F_m/K)} (b^\sigma)_{m,\mathfrak{p}_m} k_2(\sigma)(1+T)^{k_2(\sigma)} + \\ + (1+T)f'(T)((1+T)^{p^{m+1}} - 1) + p^{m+1} f(T)(1+T)^{p^{m+1}}.$$

Suppose now that for $j \leq 0$ we have

$$\left((1+T)\frac{d}{dT} \right)^{-j+1} h_b(T) = \sum_{\sigma \in Gal(F_m/K)} (b^\sigma)_{m,\mathfrak{p}_m} k_2(\sigma)^{-j+1}(1+T)^{k_2(\sigma)} + \\ + g_1(T)((1+T)^{p^{m+1}} - 1) + p^{m+1} g_2(T)$$

with $g_1(T), g_2(T) \in \hat{\mathcal{R}}_\infty[[T]]$. Applying the derivation $(1+T)d/dT$ we obtain

$$\left((1+T)\frac{d}{dT} \right)^{-j} h_b(T) = \sum_{\sigma \in Gal(F_m/K)} (b^\sigma)_{m,\mathfrak{p}_m} k_2(\sigma)^{-j}(1+T)^{k_2(\sigma)} + \\ + (1+T)g_1'(T)((1+T)^{p^{m+1}} - 1) + \\ + p^{m+1} g_1(T)(1+T)^{p^{m+1}} + p^{m+1} g_2'(T)$$

We conclude that for every $j \leq 0$ we have

$$\left((1+T)\frac{d}{dT}\right)^{-j} h_b(T) \equiv \sum_{\sigma \in Gal(F_m/K)} (b^\sigma)_{m,\mathfrak{p}_m} k_2(\sigma)^{-j} (1+T)^{k_2(\sigma)} \mod \mathfrak{p}_\infty^{m+1}.$$

□

4.1.3 Power series $g_\beta(T_1, T_2)$

In the previous section, we have studied the construction of the power series for norm-coherent system of units. We will see how these series allow us to construct p-adic measures to interpolate some L-values. In the work of Iwasawa [Iwa69] for the cyclotomic case and Coates and Wiles [CW77] for the quadratic imaginary, they constructed the theory for one variable L-function. Yager [Yag82] discovered that in the case of K quadratic imaginary, it is possible to construct a richer measure with a two variable dependence. To do this we need to enlarge the Coleman power series to a two variable type that will contain the information of the collection $(g_m)_m$.

Theorem 4.4. *For each $\beta \in U'_\infty$ there is a unique power series $g_\beta(T_1, T_2) \in \hat{\mathcal{R}}_\infty[[T_1, T_2]]$ such that*

$$g_\beta(T_1, T_2) \equiv \sum_{\sigma \in Gal(F_m/K)} (g^\sigma_{m,\beta}(T_1))_{\mathfrak{p}_m} (1+T_2)^{k_2(\sigma)} \mod ((1+T_2)^{p^{m+1}} - 1)$$

for all $m \geq 0$. Moreover, g_β satisfies the functional equation

$$\pi g_\beta([\pi]T_1, (1+T_2)^{k_2(\varphi)^{-1}} - 1) = \sum_{\eta \in \hat{E}_\pi} g_\beta(T_1[+]\eta, T_2)$$

and for every $\sigma \in G_\infty$

$$g_{\beta^\sigma}(T_1, T_2) = k_1(\sigma) g_\beta([k_1(\sigma)](T_1), (1+T_2)^{k_2(\sigma)^{-1}} - 1).$$

Proof. Recall that by Lemma 4.1.3, for $m' \geq m$ we have $g_{m,\beta} = Tr_{m',m}(g_{m',\beta})$. From the proof of Theorem 4.3 we deduce the existence and uniqueness of the power series $g_\beta(T_1, T_2) \in \hat{\mathcal{R}}_\infty[[T_1, T_2]]$.In particular for $m \geq 0$ there exists $f \in \hat{\mathcal{R}}_\infty[[T_1, T_2]]$ such that

$$g_\beta(T_1, T_2) = \sum_{\sigma \in Gal(F_m/K)} (g^\sigma_{m,\beta}(T_1))_{\mathfrak{p}_m} (1+T_2)^{k_2(\sigma)} + f(T_1, T_2)((1+T_2)^{p^{m+1}} - 1)$$

and then we have

$$g_\beta([\pi]T_1, (1+T_2)^{k_2(\varphi)^{-1}} - 1) = \sum_{\sigma \in Gal(F_m/K)} (g^\sigma_{m,\beta}([\pi]T_1))_{\mathfrak{p}_m} ((1+T_2)^{k_2(\varphi^{-1}\sigma)} + \\ + f_1(T_1, T_2)((1+T_2)^{k_2(\varphi)^{-1} p^{m+1}} - 1)$$

where $f_1(T_1, T_2) = f([\pi]T_1, (1+T_2)^{k_2(\varphi)^{-1}} - 1)$. In particular, we have

$$g_\beta([\pi]T_1, (1+T_2)^{k_2(\varphi)^{-1}} - 1) \equiv \sum_{\sigma \in Gal(F_m/K)} (g^{\sigma\varphi}_{m,\beta}([\pi]T_1))_{\mathfrak{p}_m} ((1+T_2)^{k_2(\sigma)}$$

modulo $((1+T)^{p^{m+1}}-1)$. Now, Lemma 4.1.3 shows that for all $\sigma \in Gal(F_m/K)$,

$$\pi(g^{\sigma\varphi}_{m,\beta}([\pi]T_1))_{\mathfrak{p}_m} = \sum_{\eta\in\hat{E}_\pi} (g^{\sigma}_{m,\beta}(T_1[+]\eta))_{\mathfrak{p}_m},$$

and so

$$g_\beta([\pi]T_1,(1+T_2)^{k_2(\varphi)^{-1}}-1) \equiv \sum_{\sigma\in Gal(F_m/K)}\sum_{\eta\in\hat{E}_\pi}(g^{\sigma}_{m,\beta}(T_1[+]\eta))_{\mathfrak{p}_m}((1+T_2)^{k_2(\sigma)}$$

modulo $((1+T)^{p^{m+1}}-1)$. Observe that for $\eta\in\hat{E}_\pi$ we have

$$g_\beta(T_1[+]\eta,T_2) \equiv \sum_{\sigma\in Gal(F_m/K)}(g^{\sigma}_{m,\beta}(T_1[+]\eta))_{\mathfrak{p}_m}((1+T_2)^{k_2(\sigma)}$$

modulo $((1+T)^{p^{m+1}}-1)$. The two equivalences give us the equation

$$\pi g_\beta([\pi]T_1,(1+T_2)^{k_2(\varphi)^{-1}}-1) = \sum_{\eta\in\hat{E}_\pi} g_\beta(T_1[+]\eta,T_2).$$

For the second equality, recalling Lemma 4.1.4, we observe

$$\begin{aligned} g_{\beta^\sigma}(T_1,T_2) &\equiv \sum_{\theta\in Gal(F_m/K)}(g^{\theta}_{m,\beta^\sigma}(T_1))_{\mathfrak{p}_m}(1+T_2)^{k_2(\theta)} \equiv \\ &\equiv \sum_{\theta\in Gal(F_m/K)}(k_1(\sigma)g^{\sigma\theta}_{m,\beta}([k_1(\sigma)](T_1))_{\mathfrak{p}_m}(1+T_2)^{k_2(\theta)} \equiv \\ &\equiv k_1(\sigma)\sum_{\theta\in Gal(F_m/K)}(g^{\theta}_{m,\beta^\sigma}(T_1))_{\mathfrak{p}_m}(1+T_2)^{k_2(\sigma^{-1}\theta)} \end{aligned}$$

modulo $((1+T_2)^{p^{m+1}}-1)$. Comparing the previous equation with the definition of $g^{\sigma}_{\beta}([k_1(\sigma)](T_1),(1+T_2)^{k_1(\sigma^{-1})}-1)$ we prove the equality

$$g_{\beta^\sigma}(T_1,T_2) = k_1(\sigma)g_\beta([k_1(\sigma)](T_1),(1+T_2)^{k_2(\sigma)^{-1}}-1).$$

□

Definition 4.3. *Let $k\geq 1$ and $j\leq 0$. We define for each $\beta\in U_\infty$,*

$$\delta_{k,j}(\beta) = \left[\left(\lambda'(T)^{-1}\frac{\partial}{\partial T_1}\right)^{k-1}\left((1+T_2))\frac{\partial}{\partial T_2}\right)^{-j} g_\beta(T_1,T_2)\right]_{|(T_1,T_2)=(0,0)}.$$

Lemma 4.1.6. *Let $k\geq 1$ and $j\leq 0$. Then $\delta_{k,j}$ is a homomorphism of $\mathbb{Z}_p$-modules from U_∞ to $\hat{\mathcal{R}}_\infty$ for all $\beta\in U_\infty$ and all $\sigma\in G_\infty$,*

$$\delta_{k,j}(\beta^\sigma) = k_1(\sigma)^k k_2(\sigma)^j\delta_{k,j}(\beta). \tag{4.2}$$

Let $U_\infty^{(i_1,i_2)}$ the $\mathbb{Z}_p[\Delta]$-submodule of U_∞ where Δ acts via $\chi_1^{i_1}\chi_2^{i_2}$. If $\beta\in U_\infty^{(i_1,i_2)}$, then $\delta_{k,j}(\beta)=0$ unless $(k,j)\equiv(i_1,i_2) \mod (p-1)$, and if $h(T_1,T_2)\in\Lambda$,

$$\delta_{k,j}(h(T_1,T_2)\beta) = h(u^k-1,u^j-1)\delta_{k,j}(\beta) \tag{4.3}$$

where u is the topological generator of $1+p\,\mathbb{Z}_p$.

Proof. First of all, by Theorem 4.2, for every $\beta_1, \beta_2 \in U_\infty$ we have that $c_{m,\beta_1\beta_2}(T) = c_{m,\beta_1}(T) \cdot c_{m,\beta_2(T)}$. In particular, from the definition of $g_{m,\beta}$ we get

$$g_{m,\beta_1\beta_2}(T) = g_{m,\beta_1}(T) + g_{m,\beta_2}(T)$$

and consequently

$$g_{\beta_1\beta_2}(T_1, T_2) = g_{\beta_1}(T_1, T_2) + g_{\beta_2}(T_1, T_2).$$

By linearity of the derivation operator, we conclude $\delta_{k,j}$ is a homomorphism. In order to prove equation (4.2) we use Theorem 4.4

$$\begin{aligned}
\delta_{k,j}(\beta^\sigma) &= \left[\left(\lambda'(T)^{-1}\frac{\partial}{\partial T_1}\right)^{k-1}\left((1+T_2))\frac{\partial}{\partial T_2}\right)^{-j} g_{\beta^\sigma}(T_1,T_2)\right]_{|(T_1,T_2)=(0,0)} = \\
&= \left[\left(\lambda'(T)^{-1}\frac{\partial}{\partial T_1}\right)^{k-1}\left((1+T_2))\frac{\partial}{\partial T_2}\right)^{-j} k_1(\sigma) g_\beta([k_1(\sigma)](T_1), (1+T_2)^{k_2(\sigma)^{-1}}-1)\right]_{|(T_1,T_2)=(0,0)} = \\
&= \left[k_1(\sigma)^k\left(\lambda'(T)^{-1}\frac{\partial}{\partial T_1}\right)^{k-1}\left((1+T_2))\frac{\partial}{\partial T_2}\right)^{-j} g_\beta(T_1, (1+T_2)^{k_2(\sigma)^{-1}}-1)\right]_{|([k_1(\sigma)](T_1),T_2)} = \\
&= \left[k_1(\sigma)^k k_2(\sigma)^j\left(\lambda'(T)^{-1}\frac{\partial}{\partial T_1}\right)^{k-1}\left((1+T_2))\frac{\partial}{\partial T_2}\right)^{-j} g_\beta(T_1,T_2)\right]_{|([k_1(\sigma)](T_1),(1+T_2)^{k_2(\sigma)^{-1}}-1)} =
\end{aligned}$$

from which observing that $([k_1(\sigma)](0), (1+0)^{k_2(\sigma)^{-1}} - 1) = (0,0)$ we conclude

$$\delta_{k,j}(\beta^\sigma) = k_1(\sigma)^k k_2(\sigma)^j \delta_{k,j}(\beta). \tag{4.4}$$

Let $\beta \in U_\infty^{(i_1,i_2)}$, then for every $\sigma \in \Delta$ we have $\beta^\sigma = \chi_1(\sigma)^{i_1}\chi_2(\sigma)^{i_2}\beta$. From the previous equality, we then get

$$\chi_1(\sigma)^{i_1}\chi_2(\sigma)^{i_2}\delta_{k,j}(\beta) = \delta_{k,j}(\beta^\sigma) = k_1(\sigma)k_2(\sigma)^j\delta_{k,j}(\beta)$$

where in Section 2.2.3 we defined $\chi_1 = k_{1|\Delta}$, $\chi_2 = k_{2|\Delta}$ and then $\delta_{k,j}(\beta) = 0$ unless $(k,j) \equiv (i_1, i_2) \mod (p-1)$. To conclude, recall that we have $\gamma_1, \gamma_2 \in \Gamma$ such that $(k_1,k_2)(\gamma_1) = (u,1)$ and $(k_1,k_2)(\gamma_2) = (1,u)$ with u topological generator of $(1+p\,\mathbb{Z}_p)^\times$ and Δ acts as $(1+T_i)\beta = \gamma_i\beta$ for $i = 1,2$ and $\beta \in U_\infty$. Then we have

$$\delta_{k,j}((1+T_1)^n\beta) = \delta_{k,j}(\beta^{\gamma_i^n}) = k_1(\gamma_1^n)^k k_2(\gamma_1^n)^j\delta_{k,j}(\beta) = u^{kj}\delta(k,j)(\beta)$$

and analogous for $(1+T_2)$. By linearity and continuity, we conclude

$$\delta_{k,j}(h(T_1,T_2)\beta) = h(u^k-1, u^j-1)\delta_{k,j}(\beta).$$

□

To conclude this section, the following lemma establishes the connection between the differential operators δ_j of definition 4.2 and $\delta_{k,j}$ of definition 4.3. In particular, it relates the differentials of the 1 variable power series and the two variable one.

Lemma 4.1.7. *Let $\beta \in U_\infty$ and consider*

$$d_k(\beta) = \left[\left(\lambda'(T)^{-1} d/dT \right)^{k-1} g_{m,\beta}(T) \right]_{|T=0} \in \mathcal{R}_m.$$

It satisfies the equation

$$\delta_{k,j}(\beta) = \delta_j(d_k(\beta))$$

and the congruence

$$\delta_{k,j}(\beta) \equiv \sum_{\sigma \in Gal(F_m/K)} k_2(\sigma)^{-j} (d_k(\beta)^\sigma)_{m,\mathfrak{p}_m} \mod \mathfrak{p}_\infty^{m+1}.$$

Proof. From the definition of the power series $h_b(T)$ we have

$$\begin{aligned} h_{d_k(\beta)} &\equiv \sum_{\sigma \in Gal(F_m/K)} (d_k(\beta)^\sigma)_{m,\mathfrak{p}_m} (1+T_2)^{k_2(\sigma)} \equiv \\ &\equiv \sum_{\sigma \in Gal(F_m/K)} \left[\left(\lambda'(T_1)^{-1} d/dT_1 \right)^{k-1} g^\sigma_{m,\beta,\mathfrak{p}_m} \right]_{|T_1=0} (1+T_2)^{k_2(\sigma)} \end{aligned}$$

modulo $((1+T_2)^{p^{m+1}} - 1)$. From the previous congruence and the definition of δ_j we observe $\delta_{k,j}(\beta) = \delta_j(d_k(\beta))$. By Lemma 4.1.5 we conclude

$$\delta_{k,j}(\beta) = \delta_j(d_k(\beta)) \equiv \sum_{\sigma \in Gal(F_m/K)} k_2(\sigma)^{-j} (d_k(\beta)^\sigma)_{m,\mathfrak{p}_m} \mod \mathfrak{p}_\infty^{m+1}.$$

□

4.2 Coleman power series for elliptic units

In Section 3.3 we have defined the elliptic units as special values of the theta function. In particular $C'_{n,m}$ is the subgroup of the units of $K_{n,m}$ generated by $\Theta(\varepsilon_n^{m+1}\tau_n + \rho_m; \mu)$ for all $\mu \in \mathcal{S}$. In Corollary 3.7.3 we have defined a collection of norm-coherent elliptic units

$$e_{n,m}(\mu) = \Lambda_m^{\varphi^{-n}}(z;\mu)|_{z=\varepsilon_n^{m+1}\tau_n}$$

with $(e_{n,m}(\mu)) \in U'_\infty$. Let $P_m(z) \in F_m[[z]]$ be the Laurent expansion of $\Lambda_m(\overline{\pi}^{-(m+1)}z; \mu)$. By abuse of notation we will denote

$$\Lambda_m(\overline{\pi}^{-(m+1)}\lambda(T); \mu) = P_m(\lambda(T))$$

where λ is the usual formal logarithm map. We can then finally apply the Coleman theory developed before to the system of elliptic units.

Theorem 4.5. *Let $\mu \in \mathcal{S}$. Then the Coleman power series $c_{m,e(\mu)}(T) \in \mathcal{R}_m[[T]]$ attached to $e(\mu)$ is given by*

$$c_{m,e(\mu)}(T) = \Lambda_m(\overline{\pi}^{-(m+1)}\lambda(T); \mu) \tag{4.5}$$

where $\lambda(T)$ is the formal logarithm of $\hat{E}$.

Proof. First of all, recall that by construction we have

$$[\overline{\pi}^{-(m+1)}]u_n = \varepsilon(\varepsilon_n^{m+1}\tau_n)$$

then it follows

$$\Lambda_m^{\varphi^{-n}}(\overline{\pi}^{-(m+1)}\lambda(u_n);\mu) = \Lambda_m^{\varphi^{-n}}(z;\mu)|_{z=\varepsilon_n^{m+1}\tau_n} = e_{n,m}(\mu)$$

since λ is the inverse of the formal exponential map ε. By the uniqueness of the Coleman power series, we conclude the proof. □

By the definition of the logarithmic derivative of the Coleman power series 4.1 we then have

$$g_{m,e(\mu)}(T) = \lambda'(T)^{-1}\frac{d}{dT}\log\Lambda_m(\overline{\pi}^{-(m+1)}\lambda(T);\mu).$$

Furthermore, the two variable function $g_{e(\mu)}(T_1,T_2)$ is then given by

$$g_{e(\mu)}(T_1,T_2) \equiv \sum_{\sigma\in Gal(F_m/K)} (g^{\sigma}_{m,e(\mu)}(T_1))_{\mathfrak{p}_m}(1+T_2)^{k_2(\sigma)} \mod ((1+T_2)^{p^{m+1}}-1)$$

for all $m \geq 0$.

Recall that if $b \in \varprojlim \mathcal{R}_m$ and $j \leq 0$ we have defined the two differential operators

$$\delta_j(b) = \left((1+T)\frac{d}{dT}\right)^{-j} h_b(T)|_{T=0} \in \hat{\mathcal{R}}_\infty,$$
$$\delta_{k,j}(e(\mu)) = \left[\left(\lambda'(T)^{-1}\frac{\partial}{\partial T_1}\right)^{k-1}\left((1+T_2))\frac{\partial}{\partial T_2}\right)^{-j} g_{e(\mu)}(T_1,T_2)\right]_{|(T_1,T_2)=(0,0)}.$$

Prop. 4.2.1. *There exists an isomorphism of formal groups defined over* $\hat{\mathcal{R}}_\infty$,

$$\eta : \hat{E} \to \hat{\mathbb{G}}_m$$
$$S \mapsto \eta(S) = \Omega_{\mathfrak{p}}S + \cdots \in \hat{\mathcal{R}}_\infty[[S]]$$

satisfying

$$1+\eta\left(\varepsilon\left(\frac{\Omega_\infty}{\pi^{n+1}}\right)\right) = \left(\frac{\Omega_\infty}{\pi^{n+1}}, \varepsilon_n^{-n+1}\frac{\Omega_\infty}{\overline{\pi}^{n+1}}\right)_n \tag{4.6}$$

where $(,)_n$ *denotes the Weil pairing of the* p^{n+1}*-division points of* L*. In particular,* $\Omega_{\mathfrak{p}}$ *is a unit in* $\hat{\mathcal{R}}_\infty$ *and uniquely determined by the choice of the embedding of the fields* $K_{n,m}$ *in* $\Xi_{n,m}$*.*

Proof. First of all, since $\hat{E}$ is a formal group of height 1, Lubin [Lub64] proved that there exists an isomorphism between $\hat{E}$ and $\hat{\mathbb{G}}_m$. In particular, Tate [Tat67] has shown that we have a natural isomorphism between the isomorphism group of p-divisible groups and their Tate-modules

$$Hom(\hat{E},\hat{\mathbb{G}}_m) \xrightarrow{\sim} Hom(\varprojlim E[\pi^{n+1}], \varprojlim \mu_{p^{n+1}}).$$

The Weil pairing shows that $Hom(\varprojlim E[\pi^{n+1}], \varprojlim \mu_{p^{n+1}})$ is naturally isomorphic to $\varprojlim E[\overline{\pi}^{n+1}]$. Recall that we fixed $\varepsilon_n \in \mathcal{O}_K$ such that $\varepsilon_n\overline{\pi} \equiv 1 \mod \mathfrak{p}^{n+1}$. Then we consider the isomorphism $\eta : \hat{E} \to \hat{\mathbb{G}}_m$ associated to $(\varepsilon_n^{-n+1}\Omega_\infty/\overline{\pi}^{n+1})$ obtaining

$$1+\eta\left(\varepsilon\left(\frac{\Omega_\infty}{\pi^{n+1}}\right)\right) = \left(\frac{\Omega_\infty}{\pi^{n+1}}, \varepsilon_n^{-n+1}\frac{\Omega_\infty}{\overline{\pi}^{n+1}}\right)_n.$$

Since $\eta(S) = \Omega_{\mathfrak{p}} S + \cdots$ is an isomorphism, then we have $\Omega_{\mathfrak{p}}$ is a unit in $\hat{\mathcal{R}}_\infty$. In particular, since $\hat{\mathbb{G}}_m$ is isomorphic to $\hat{\mathbb{G}}_a$ we deduce by definition of the logarithm map λ that η has a series expansion of the form

$$\eta(S) = \exp(\Omega_{\mathfrak{p}} \lambda(S)) - 1.$$

□

Theorem 4.6. *Let $\mu \in \mathcal{S}$ and let k, j be integers such that $k > -j \geq 0$. Then*

$$\begin{aligned}\delta_{k,j}(\langle e(\mu)\rangle) =& 12(-1)^{k+1-j}(k-1)! f^k \sum_{\mathfrak{a} \in I} \mu(\mathfrak{a})(N\mathfrak{a} - \psi^k(\mathfrak{a})\overline{\psi}^j(\mathfrak{a})) \\ & \left(1 - \frac{\overline{\psi}^{k-j}(\overline{\mathfrak{p}})}{N\overline{\mathfrak{p}}^k}\right)\left(\frac{2\pi}{\sqrt{d_K}}\right)^{-j} \Omega_{\mathfrak{p}}^j \Omega_\infty^{j-k} L(\overline{\psi}^{k-j}, k).\end{aligned}$$

Proof. The proof is based on Katz formulae [Kat76] and can be read in full detail in Yager's paper [Yag82] Theorem 15. □

CHAPTER 5

p-adic Interpolation

5.1 Two variable p-adic measures

5.1.1 Basic results on power series

Lemma 5.1.1 (Division algorithm). *Let A a complete valuation ring with $\mathfrak{m}$ maximal ideal and residue field of characteristic p. Let $\Lambda = A\left[[T_1, \cdots, T_n]\right]$, $g \in \Lambda$ and $f_i \in A\left[[T_i]\right]$ such that $f_i \notin \mathfrak{m}A\left[[T_i]\right]$ for $i = 1, \ldots, n$. Let m_i the largest integer such that $f_i \in \mathfrak{m}\Lambda + (T_i^{m_i})$. Then*

$$g = q_1 f_1 + \cdots q_n f_n + r \tag{5.1}$$

with $q_i \in \Lambda$ and $r \in A\left[T_1, \ldots, T_n\right]$.

Proof. Let $u_i \in A^\times$ the coefficient of $T_i^{m_i}$ in f_i i.e. f_i is of the form $f_i = a_i T_i^{m_i} + b_i$ with $b_i \in \mathfrak{m}A[T_1, \ldots, T_n]$ and $a_i \in \Lambda$ such that $a_i \equiv u \mod \deg 1$. In particular, we have $a_i \in \Lambda^\times$. Let $q'_{0,i} \in \Lambda$ and $r_0 \in A[T_1, \ldots, T_n]$ such that

$$g = q'_{0,1} T_1^{m_i} + \cdots + q'_{0,n} T_n^{m_n} + r_0.$$

Taking $q_{0,i} = a_i^{-1} q'_{0,i}$ we can rewrite

$$g = q_{0,1} a_1 T_1^{m_i} + \cdots + q_{0,n} a_n T_n^{m_n} + r_0.$$

Since $f_i \equiv a_i T_i \mod \mathfrak{m}\Lambda$ we deduce

$$g \equiv q_{0,1} f_1 + \cdots + q_{0,n} f_n + r_0 \mod \mathfrak{m}\Lambda.$$

Consider now $g_1 = g - q_{0,1} f_1 - \cdots + q_{0,n} f_n - r_0 \in \mathfrak{m}\Lambda$ and repeat the procedure finding $q_{1,i} \in \Lambda$ such that

$$g_1 \equiv q_{1,1} f_1 + \cdots + q_{1,n} f_n + r_1 \mod \mathfrak{m}^2\Lambda.$$

In this way, we obtain the following congruence

$$g \equiv (q_{0,1} + q_{1,1}) f_1 + \cdots + (q_{n,1} + q_{n,1}) f_n + (r_0 + r_1) \mod \mathfrak{m}^2\Lambda.$$

Iterating this argument we define $q_i = q_{0,i} + q_{1,i} + \cdots$, $r = r_0 + r_1 + \cdots$ for $i = 1, \ldots, n$ that satisfy the equality

$$g = q_1 f_1 + \cdots + q_n f_n + r.$$

Observe that by construction we have that the maximal exponent of T_i in r is less than m_i. □

Definition 5.1. *A distinguished polynomial $p \in A[T]$ is a polynomial with leading coefficient 1 and such that $p \equiv T^{\deg p} \mod \mathfrak{m}A[T]$.*

Theorem 5.1 (Weierstrass preparation). *Let $g \in A[[T]]$ be a power series such that $g \notin \mathfrak{m}A[[T]]$. Then there exists a unique distinguished polynomial $p \in A[T]$ and a unit $u \in A[[T]]^\times$ such that*

$$g = up.$$

Proof. Let $n \geq 0$ be an integer such that $g \in \mathfrak{m}A[[T]] + (T^n)$ and denote $u_0 \in A[[T]]^\times$ the coefficient of T^n in g. Applying the division algorithm, we obtain a unique $q \in A[[T]]$ and $r \in \mathfrak{m}A[T]$ with $\deg r < n$ such that

$$T^n = qg + r.$$

Consider $f = T^n - r$ and observe f is a distinguished polynomial. Furthermore, $qu_0 \equiv 1 \mod \deg T$ and then q is a unit in $A[[T]]$. We conclude

$$g = (T^n - r)q^{-1} = fu.$$

□

Corollary 5.1.1. *Let A be a PID ring. Then $A[[T]]$ is a unique factorization domain.*

Proof. Consider $\pi \in A$ such that $\mathfrak{m} = (\pi)$. For every $g \in A[[T]]$ there exists $n \geq 0$ such that $g\pi^{-n} \notin \mathfrak{m}A[[T]]$. By Weierstrass preparation theorem there exists $p \in A[T]$ and $u \in A[[T]]^\times$ such that

$$g = \pi^n pu.$$

By the uniqueness of the factorization of p in $A[T]$ we conclude that g uniquely decomposes in $A[[T]]$. □

Lemma 5.1.2. *Suppose that a p prime number lies in $\mathfrak{m}$. Let $h \in A[[T_1, T_2]]$ be a power series, then for every $n \geq 0$ there exists $b_{k,j} \in A$ for $k, j = 1, \ldots p^n - 1$ such that we have a unique decomposition*

$$h(T_1, T_2) \equiv \sum_{k,j=1}^{p^n-1} b_{k,j}(1 + T_1)^k (1 + T_2)^j \mod ((1 + T_1)^{p^n} - 1, (1 + T_2)^{p^n} - 1).$$

Proof. Observe that $((1 + T_i)^{p^n} - 1) \in \mathfrak{m}A[[T_i]] + (T_i^{p^n})$, then by the division algorithm we can write

$$h(T_1, T_2) = q_1(T_1, T_2)((1 + T_1)^{p^n} - 1) + q_2(T_1, T_2)((1 + T_2)^{p^n} - 1) + r(T_1, T_2)$$

for $q_i \in A[[T_i]]$ and $r \in A[T_1, T_2]$ of degree less then n in both $T_i's$. Define $b_{k,j} \in A$ to be the coefficients of the polynomial $r(T_1 - 1, T_2 - 1)$

$$r(T_1 - 1, T_2 - 1) = \sum_{k,j=0}^{p^n-1} b_{k,j} T_1^k T_2^j.$$

We then conclude

$$h(T_1, T_2) \equiv r(T_1, T_2) \equiv \sum_{k,j=1}^{p^n-1} b_{k,j}(1 + T_1)^k (1 + T_2)^j \mod ((1 + T_1)^{p^n} - 1, (1 + T_2)^{p^n} - 1).$$

□

5.1.2 Γ-transform

Definition 5.2. *Let $k \geq 0$ be an integer and define the binomial coefficient function $\binom{x}{k} : \mathbb{Z}_p \to \mathbb{Z}_p$ to be*

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

Definition 5.3. *Let μ be a measure on $\mathbb{Z}_p \times \mathbb{Z}_p$ with values in A. Then we define the power series $f_\mu \in A[[T_1, T_2]]$ associated to μ as follows*

$$f_\mu(T_1, T_2) = \sum_{n,m \geq 0} \left(\int_{\mathbb{Z}_p^2} \binom{x_1}{n} \binom{x_2}{m} d\mu(x_1, x_2) \right) T_1^n T_2^m =$$
$$= \int_{\mathbb{Z}_p^2} (1 + T_1)^{x_1} (1 + T_2)^{x_2} d\mu(x_1, x_2) \tag{5.2}$$

Conversely, given $f \in A[[T_1, T_2]]$ we would like to associate a unique A-valued measure on μ_f to which it corresponds under equation 5.2. By Lemma 5.1.2 we can write

$$f(T_1, T_2) \equiv \sum_{k,j=0}^{p^n-1} b_{k,j}(1 + T_1)^k (1 + T_2)^j \mod ((1 + T_1)^{p^n} - 1, (1 + T_2)^{p^n} - 1).$$

The following Lemma will give the construction for the measure μ_f.

Lemma 5.1.3. *Let $f \in A[[T_1, T_2]]$ be a power series. Then there exists a unique measure μ for which*

$$\int_{(k+p^n \mathbb{Z}_p)\times(j+p^n \mathbb{Z}_p)} d\mu = b_{k,j}. \tag{5.3}$$

In particular, we obtain $f_\mu = f$.

Proof. In order to define a measure μ from a map from the open sets $(k + p^n \mathbb{Z}_p) \times (j + p^n \mathbb{Z}_p)$ to $\mathbb{Z}_p$ to A we need to verify that it satisfies the distribution relation

$$\mu((k + p^n \mathbb{Z}_p) \times (j + p^n \mathbb{Z}_p)) = \sum_{r,s=0}^{p-1} \mu((k + rp^n + p^{n+1} \mathbb{Z}_p) \times (j + sp^n + p^{n+1} \mathbb{Z}_p)).$$

Comparing the two decompositions $\mod p^n \mod p^{n+1}$

$$f(T_1, T_2) \equiv \sum_{k,j=1}^{p^n-1} b_{k,j}(1 + T_1)^k (1 + T_2)^j \mod ((1 + T_1)^{p^n} - 1, (1 + T_2)^{p^n} - 1)$$

$$f(T_1,T_2) \equiv \sum_{k,j=1}^{p^{n+1}-1} c_{k,j}(1+T_1)^k(1+T_2)^j \mod ((1+T_1)^{p^{n+1}}-1,(1+T_2)^{p^{n+1}}-1)$$

we obtain the relation

$$b_{k,j}(1+T_1)^k(1+T_2)^j = \sum_{r,s=0}^{p-1} c_{k+rp^n,j+sp^n}(1+T_1)^{k+rp^n}(1+T_2)^{j+sp^n}$$

and in particular

$$b_{k,j} = \sum_{r,s=0}^{p-1} c_{k+rp^n,j+sp^n}.$$

We conclude μ is an A-valued measure on $\mathbb{Z}_p^2$. We now need to show that $f = \mu_f$. From the congruence of Lemma 5.1.2 we have

$$\begin{aligned} f(T_1,T_2) &\equiv \sum_{k,j=1}^{p^n-1} b_{k,j}(1+T_1)^k(1+T_2)^j \equiv \sum_{k,j=1}^{p^n-1} b_{k,j}\sum_{r=0}^{k}\sum_{s=0}^{j}\binom{k}{r}\binom{j}{s}T_1^rT_2^s \equiv \\ &\equiv \sum_{k,j=1}^{p^n-1}\sum_{r=0}^{p^n-1}\sum_{s=0}^{p^n-1} b_{k,j}\binom{k}{r}\binom{j}{s}T_1^rT_2^s \mod ((1+T_1)^{p^{n+1}}-1,(1+T_2)^{p^{n+1}}-1). \end{aligned}$$

On the other hand, recall that we can compute the value of the integral of a function φ on $\mathbb{Z}_p^2$ using the Riemann sums

$$\int_{\mathbb{Z}_p^2}\varphi d\mu = \lim_{n\to\infty}\sum_{(k+p^m\,\mathbb{Z}_p)\times(j+p^n\,\mathbb{Z}_p)}\varphi(k,j)\mu((k+p^n\,\mathbb{Z}_p)\times(j+p^n\,\mathbb{Z}_p)).$$

In particular we obtain

$$\int_{\mathbb{Z}_p^2}\binom{x_1}{n}\binom{x_2}{m}d\mu = \lim_{n\to\infty}\sum_{(k+p^n\,\mathbb{Z}_p)\times(j+p^n\,\mathbb{Z}_p)}\binom{k}{r}\binom{j}{s}b_{k,j}.$$

We then conclude $f_\mu = f$. □

Definition 5.4. *Let x a unit in $\mathbb{Z}_p$, we write $x = \omega(x)\langle x\rangle$, where $\omega(x)$ is the Teichmuller character associated to x and $\langle x\rangle \equiv 1 \mod p$. Consider $(i_1,i_2) \in (\mathbb{Z}/(p-1)\,\mathbb{Z})\times(\mathbb{Z}/(p-1)\,\mathbb{Z})$ and f a power series in $A[[T]]$ corresponding to a measure μ_f. We define the Γ-transform*

$$\Gamma_f^{(i_1,i_2)} : \mathbb{Z}_p^2 \to A$$

by

$$\Gamma_f^{(i_1,i_2)}(s_1,s_2) = \int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}\omega^{i_1}(x_1)\omega^{i_2}(x_2)d\mu_f. \tag{5.4}$$

Definition 5.5. *Let u be a topological generator of $1+p\,\mathbb{Z}_p$, then we can define a homomorphism $l : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ such that*

$$\langle x\rangle = u^{l(x)}$$

for every $x \in \mathbb{Z}_p^\times$.

Lemma 5.1.4. *Let $f \in A[[T_1,T_2]]$ be a power series and take $(i_1,i_2) \in \mathbb{Z}/(p-1)\,\mathbb{Z}$. Then there exists $f^{(i_1,i_2)} \in A[[T_1,T_2]]$ such that for all $s_1,s_2 \in \mathbb{Z}_p$ we have*

$$\Gamma_f^{(i_1,i_2)}(s_1,s_2) = f^{(i_1,i_2)}(u^{s_1}-1,u^{s_2}-1). \tag{5.5}$$

Proof. By definition of l homomorphism and Γ-transform we have

$$\Gamma_f^{(i_1,i_2)}(s_1,s_2) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (1+u^{s_1}-1)^{l(x_1)}(1+u^{s_2}-1)^{l(x_2)}\omega^{i_1}(x_1)\omega^{i_2}(x)d\mu_f.$$

Considering the binomial expansion of the right side we obtain

$$\sum_{n,m\geq 0}(u^{s_1}-1)^n(u^{s_2}-1)^m \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \binom{l(x_1)}{n}\binom{l(x_2)}{m}\omega^{i_1}(x_1)\omega^{i_2}(x_2)d\mu_f.$$

We can then define $f^{(i_1,i_2)} \in A[[T_1,T_2]]$ taking the coefficients of $T_1^nT_2^m$ to be

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \binom{l(x_1)}{n}\binom{l(x_2)}{m}\omega^{i_1}(x_1)\omega^{i_2}(x_2)d\mu_f.$$

□

On the ring of power series $A[[T_1,T_2]]$, we can consider the operator D_i defined by $(1+T_i)\partial/\partial T_i$. These operators will play a fundamental role in the interpolation property.

Lemma 5.1.5. *Let μ be a measure on $\mathbb{Z}_p^2$. Then, for $n,m \geq 0$ consider the measures $\mu_{n,m}$ corresponding to the power series $D_1^nD_2^m f_\mu \in A[[T_1,T_2]]$. For all measurable functions $\varphi : \mathbb{Z}_p^2 \to A$ we have the following identity*

$$\int_{\mathbb{Z}_p^2} \varphi(x_1,x_2)d\mu_{n,m}(x_1,x_2) = \int_{\mathbb{Z}_p^2} \varphi(x_1,x_2)x_1^nx_2^m d\mu.$$

Proof. We proceed by induction. First of all, consider the measure $\mu_{1,0}$ associated with the power series $D_1 f\mu$. Explicitly we have $a_{k,j} \in A$ such that

$$f_\mu(T_1,T_2) = \sum_{k,j\geq 0} a_{k,j}T_1^kT_2^j$$

and applying D_1 operator we get

$$D_1 f\mu(T_1,T_2) = (1+T_1)\left(\sum_{k,j\geq 0} k a_{k,j}T_1^{k-1}T_2^j\right) = \sum_{k,j\geq 0}(ka_{k,j}+(k+1)a_{k+1,j})T_1^kT_2^j.$$

From equation 5.2 applied to f_μ and $D_1 f_\mu$ we obtain

$$f_\mu(T_1,T_2) = \sum_{k,j\geq 0}\left(\int_{\mathbb{Z}_p^2}\binom{x_1}{k}\binom{x_2}{j}d\mu\right)T_1^kT_2^j$$
$$D_1 f_\mu(T_1,T_2) = \sum_{k,j\geq 0}\left(\int_{\mathbb{Z}_p^2}\binom{x_1}{k}\binom{x_2}{j}d\mu_{1,0}\right)T_1^kT_2^j$$

and then by previous computation

$$\int_{\mathbb{Z}_p^2}\binom{x_1}{k}\binom{x_2}{j}d\mu_{1,0} = k\left(\int_{\mathbb{Z}_p^2}\binom{x_1}{k}\binom{x_2}{j}d\mu\right) + (k+1)\left(\int_{\mathbb{Z}_p^2}\binom{x_1}{k+1}\binom{x_2}{j}d\mu\right) =$$
$$= \int_{\mathbb{Z}_p^2}\left(k\binom{x_1}{k} + (k+1)\binom{x_1}{k+1}\right)\binom{x_2}{j}d\mu.$$

From the straightforward identity

$$k\binom{x_1}{k} + (k+1)\binom{x_1}{k+1} = x_1\binom{x_1}{k}$$

we conclude

$$\int_{\mathbb{Z}_p^2} \binom{x_1}{k}\binom{x_2}{j} d\mu_{1,0} = \int_{\mathbb{Z}_p^2} \binom{x_1}{k}\binom{x_2}{j} x_1 d\mu.$$

Clearly, the case $D_2 f_\mu$ is completely symmetric. Consider now $n \geq 1$, $m \geq 0$ and suppose the Lemma holds for $n-1$. Let $g = D_1^{n-1} D_2^m f_\mu$, $\nu = \mu_{n-1,m}$ then we have $D_1 g = D_1^n D_2^m f$ and

$$\int_{\mathbb{Z}_p^2} \varphi \nu_{1,0} = \int_{\mathbb{Z}_p^2} \varphi x_1 \nu = \int_{\mathbb{Z}_p^2} (\varphi x_1)\, x_1^{n-1} x_2^m \mu = \int_{\mathbb{Z}_p^2} \varphi \mu_{n,m}.$$

In particular we have $\nu_{1,0} = \mu_{n,m}$. Applying the first step for $n = 1$ we conclude $D_1 g$ corresponds to $\nu_{1,0}$ and then $D_1^n D_2^m f_\mu$ corresponds to $\mu_{n,m}$. The case $m \geq 1$ is analogous. □

Recall that a measure μ is supported on a measurable subset B of $\mathbb{Z}_p^2$ if, for all measurable functions $\varphi : \mathbb{Z}_p^2 \to A$ we have

$$\int_{\mathbb{Z}_p^2} \varphi\mu = \int_{\mathbb{Z}_p^2} \varphi \mathbb{1}_B \mu = \int_B \varphi\mu.$$

Lemma 5.1.6. *Suppose $f \in A[[T_1, T_2]]$ a power series corresponding to a measure μ_f supported on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. Let $(i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$ be a pair of integers modulo $p-1$. Then, for each pair of integers $k_1, k_2 \geq 0$ such that $(k_1, k_2) \equiv (i_1, i_2) \mod (p-1)$ we have*

$$\Gamma_f^{(i_1,i_2)}(k_1, k_2) = (D_1^{k_1} D_2^{k_2} f)(0, 0). \tag{5.6}$$

Proof. From the definition of the Teichmuller character, we have that for all $(x_1, x_2) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$

$$x_1^{k_1} x_2^{k_2} = \langle x_1 \rangle^{k_1} \langle x_2 \rangle^{k_2} \omega(x_1)^{i_1} \omega(x_2)^{i_2}.$$

From the definition of the Γ-transform (5.4) we have

$$\Gamma_f^{(i_1,i_2)}(k_1, k_2) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x_1^{k_1} x_2^{k_2} d\mu_f$$

and then in particular, using the previous Lemma, we deduce

$$\Gamma_f^{(i_1,i_2)}(k_1, k_2) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \binom{x_1}{0}\binom{x_2}{0} d\mu_{D_1^{k_1} D_2^{k_2} f}$$

where we have used the fact that the binomial function in 0 is the constant 1. From Definition 5.2 of the associated power series to a measure, we conclude $\Gamma_f^{(i_1,i_2)}(k_1, k_2) = (D_1^{k_1} D_2^{k_2})(0, 0)$. □

We conclude the section by giving the construction of the power series corresponding to the restriction of a measure to $\mathbb{Z}_p^\times \times \mathbb{Z}_p$.

Lemma 5.1.7. *Let $f \in A[[T_1, T_2]]$ be a power series and consider*

$$\tilde{f}(T_1, T_2) := f(T_1, T_2) - \frac{1}{p} \sum_{\zeta^p = 1} f(\zeta(1 + T_1) - 1, T_2)$$

where the sum on the right is taken over the set of all the p-roots of unity on A. Then $\tilde{f} \in A[[T_1, T_2]]$ and for all measurable functions $\varphi : \mathbb{Z}_p^2 \to A$,

$$\int_{\mathbb{Z}_p^2} \varphi \mu_{\tilde{f}} = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \varphi \mu_f.$$

Proof. First of all observe that for $n \geq 0$ and $g(T_1, T_2) = (1+T_1)^{p_n} - 1$ we have

$$g(\zeta(1+T_1) - 1, T_2) = (1+T_1)^{p_n} - 1 = g(T_1, T_2).$$

In particular, by Lemma 5.1.2 we have

$$f(T_1, T_2) \equiv \sum_{k,j=1}^{p^n-1} b_{k,j}(1+T_1)^k(1+T_2)^j \mod ((1+T_1)^{p^n} - 1, (1+T_2)^{p^n} - 1)$$

and then

$$f(\zeta(1+T_1) - 1, T_2) \equiv \sum_{k,j=1}^{p^n-1} b_{k,j}\zeta^k(1+T_1)^k(1+T_2)^j \mod ((1+T_1)^{p^n} - 1, (1+T_2)^{p^n} - 1).$$

From the fact that $\sum_\zeta \zeta^k$ is 0 if $(k,p) = 1$ and 1 otherwise, we deduce that in the decomposition

$$\tilde{f}(T_1, T_2) \equiv \sum_{k,j=1}^{p^n-1} c_{k,j}(1+T_1)^k(1+T_2)^j \mod ((1+T_1)^{p^n} - 1, (1+T_2)^{p^n} - 1)$$

the coefficients $c_{k,j}$ are

$$c_{k,j} = \begin{cases} b_{k,j} & \text{if } (k,p) = 1 \\ 0 & \text{if } (k,p) \neq 1. \end{cases}$$

From the Lemma 5.1.3 we conclude that the measure $\mu_{\tilde{f}}$ is supported on $\mathbb{Z}_p^\times \times \mathbb{Z}_p$. □

5.2 Construction of $\mathcal{G}_\beta^{(i_1,i_2)}$

We will denote $\iota(T) \in \hat{\mathcal{R}}_\infty$ the inverse of $\eta(T)$. Recall that $g_\beta(T_1, T_2) \in \hat{\mathcal{R}}_\infty$ denotes the unique two variable power series attached to an element $\beta \in U_\infty$ defined in Theorem 4.4. Then we have the following result.

Lemma 5.2.1. *Let $\beta \in U_\infty$ and consider $h_\beta(T_1, T_2) = g_\beta(\iota(T_1), T_2) \in \hat{\mathcal{R}}_\infty$. The $\hat{\mathcal{R}}_\infty$-valued measure on $\mathbb{Z}_p^2$ corresponding to $h_\beta(T_1, T_2)$ is supported on $\mathbb{Z}_p \times \mathbb{Z}_p^\times$.*

Proof. By Theorem 4.4 we have

$$h_\beta(T_1, T_2) \equiv \sum_{\sigma \in Gal(F_m/K)} (g_{m,\beta}^\sigma(\iota(T_1)))_{\mathfrak{p}_m}(1+T_2)^{k_2(\sigma)} \mod ((1+T_2)^{p^{m+1}} - 1).$$

Since k_2 take values in $\mathbb{Z}_p^\times$, then by Lemma 5.1.3 we conclude $h_\beta(T_1, T_2)$ corresponds to a measure supported on $\mathbb{Z}_p \times \mathbb{Z}_p^\times$. □

Recall that by Lemma 5.1.7 we have that

$$\tilde{h}_\beta(T_1, T_2) = h_\beta(T_1, T_2) - \frac{1}{p}\sum_{\zeta^p=1} h_\beta(\zeta(1+T_1)-1, T_2)$$

gives rise to a measure supported on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$.

Lemma 5.2.2. *Let $k > -j \geq 0$. For each $\beta \in U_\infty$ consider $\tilde{h}_\beta(T_1, T_2) \in \hat{\mathcal{R}}_\infty[[T_1, T_2]]$. Then we have that*

$$D_1^{k-1} D_2^{-j} \tilde{h}_\beta(T_1, T_2)|_{(0,0)} = \Omega_{\mathfrak{p}}^{1-k}\left(1 - \frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)\delta_{k,j}(\beta).$$

Proof. By construction we have $\iota \circ \eta(T) = T$ and $\eta(T) = \exp(\Omega_{\mathfrak{p}}\lambda(T)) - 1$. Then we deduce $\eta'(T) = \exp(\Omega_{\mathfrak{p}}\lambda(T))\Omega_{\mathfrak{p}}\lambda'(T)$ and so

$$\begin{aligned} 1 &= \iota'(\eta(T))\eta'(T) = \iota'(\eta(T))(1+\eta(T))\Omega_{\mathfrak{p}}\lambda'(T) \\ \iota'(\eta(T))(1+\eta(T)) &= (\Omega_{\mathfrak{p}}\lambda'(T))^{-1}. \end{aligned}$$

From this, it follows that

$$(1+T_1)\frac{d}{dT_1} f(T_1) = \left[(\Omega_{\mathfrak{p}}\lambda'(T))^{-1}\frac{d}{dT} f(T)\right]|_{T=\iota(T_1)}$$

and in particular

$$D_1^{k-1} D_2^{-j} \tilde{h}_\beta(T_1, T_2)|_{(0,0)} = \left(\Omega_{\mathfrak{p}}\lambda'(T))^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}\tilde{h}_\beta(\eta(T), T_2)|_{(0,0)}. \tag{5.7}$$

Recall that by definition of $\tilde{h}_\beta$ we have

$$\tilde{h}_\beta(\eta(T), T_2) = h_\beta(\eta(T), T_2) - \frac{1}{p}\sum_{\zeta^p=1} h_\beta(\zeta(1+\eta(T))-1, T_2).$$

In particular, since $\zeta - 1$ is a point of order p on $\mathbb{G}_m$ and ι is an isomorphism then $\iota(\zeta-1)$ runs over the solution the elements of $\hat{E}_\pi$ as ζ runs over the solution set of $\zeta^p = 1$. Moreover, we also have that

$$\eta(\iota(\zeta-1)[+]T) = (\zeta-1) + \eta(T) + (\zeta-1)\eta(T) = \zeta(1+\eta(T)) - 1$$

and so

$$h_\beta(\zeta(1+\eta(T))-1, T_2) = h_\beta(\eta(\iota(\zeta-1)[+]T), T_2) = g_\beta(\iota(\zeta-1)[+]T, T_2).$$

We conclude that

$$\tilde{h}_\beta(\eta(T), T_2) = g_\beta(T, T_2) - \frac{1}{p}\sum_{\rho\in\hat{E}_\pi} g_\beta(T[+]\rho, T_2).$$

Using the functional equation of $g_\beta(T_1, T_2)$ described in Theorem 4.4 we can rewrite the previous equation

$$\tilde{h}_\beta(\eta(T), T_2) = g_\beta(T, T_2) - \frac{\pi}{p} g_\beta([\pi]T_1, (1+T_2)^{k_2(\varphi)^{-1}} - 1).$$

Recall that the Frobenius elements φ acts on $\hat{E}_{\overline{\pi}^\infty}$ via $\psi(\mathfrak{p})$ by definition of the Hecke character then $k_2(\varphi) = \overline{\pi}$ by Lemma 2.2.6. Notice that we have

$$(1+T)\frac{d}{dT} f((1+T)^{\overline{\pi}^{-1}} - 1) = \overline{\pi}^{-1}\left((1+W)\frac{d}{dW} f(W)\right)|_{W=(1+T)^{\overline{\pi}^{-1}}-1}.$$

Then combining all these facts we obtain that equation (5.7) becomes

$$\begin{aligned}
&D_1^{k-1}D_2^{-j}\tilde{h}_\beta(T_1,T_2)|_{(0,0)} = \\
=&\left((\Omega_\mathfrak{p}\lambda'(T))^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}\left(g_\beta(T,T_2)-\frac{\pi}{p}g_\beta([\pi]T_1,(1+T_2)^{\overline{\pi}^{-1}}-1)\right)|_{(0,0)} = \\
=&\Omega_\mathfrak{p}^{1-k}\left(\lambda'(T)^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}g_\beta(T,T_2)|_{(0,0)} - \\
&-\Omega_\mathfrak{p}^{1-k}\left(\frac{\pi}{p}\right)^{k-1-j}\left(\lambda'(T)^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}\left(g_\beta([\pi]T_1,(1+T_2)^{\overline{\pi}^{-1}}-1)\right)|_{(0,0)}.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&\left(\lambda'(T)^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}\left(g_\beta([\pi]T_1,(1+T_2)^{\overline{\pi}^{-1}}-1)\right)|_{(0,0)} = \\
=&\pi^{k-1}\left(\lambda'(T)^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}g_\beta(T_1,(1+T_2)^{\overline{\pi}^{-1}}-1)|_{(0,0)} = \\
=&\pi^{k-1}\overline{\pi}^{j}\left(\lambda'(T)^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}g_\beta(T_1,T_2)|_{(0,0)}
\end{aligned}$$

that recombined together gives us

$$\begin{aligned}
&D_1^{k-1}D_2^{-j}\tilde{h}_\beta(T_1,T_2)|_{(0,0)} = \\
=&\left(1-\frac{\pi^{k-j}}{p^{1-j}}\right)\Omega_\mathfrak{p}^{1-k}\left(\lambda'(T)^{-1}\frac{\partial}{\partial T}\right)^{k-1} D_2^{-j}g_\beta(T,T_2)|_{(0,0)}.
\end{aligned}$$

By Definition 4.3 of δ_k, j we conclude

$$D_1^{k-1}D_2^{-j}\tilde{h}_\beta(T_1,T_2)|_{(0,0)} = \left(1-\frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)\Omega_\mathfrak{p}^{1-k}\delta_{k,j}(\beta)$$

□

In the case of the norm-coherent elliptic units $e(\mu)$, using the results of Section 4.2 we then have

$$\begin{aligned}
D_1^{k-1}D_2^{-j}\tilde{h}_{e(\mu)}(T_1,T_2)|_{(0,0)} =& \left(1-\frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)12(-1)^{k+1-j}(k-1)!f^k\sum_{\mathfrak{a}\in I}\mu(\mathfrak{a})(N\mathfrak{a}-\psi^k(\mathfrak{a})\overline{\psi}^j(\mathfrak{a})) \\
&\left(1-\frac{\overline{\psi}^{k-j}(\overline{\mathfrak{p}})}{N\overline{\mathfrak{p}}^k}\right)\left(\frac{2\pi}{\sqrt{d_K}}\right)^{1-j-k}\Omega_\mathfrak{p}^j\Omega_\infty^{j-k}L(\overline{\psi}^{k-j},k)
\end{aligned}$$

Theorem 5.2. *Let i_1, i_2 be integers modulo $(p-1)$ and let $\beta \in U_\infty$. Then there is a unique power series $\mathcal{G}_\beta^{(i_1,i_2)}(T_1,T_2) \in \hat{\mathcal{R}}_\infty[[T_1,T_2]]$ such that for all $k > -j \geq 0$ satisfying $(k,j) \equiv (i_1,i_2) \mod (p-1)$,*

$$\mathcal{G}_\beta^{(i_1,i_2)}(u^k-1,u^j-1) = \left(1-\frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)\Omega_\mathfrak{p}^{1-k}\delta_{k,j}(\beta).$$

Moreover, if $h \in \Lambda$,

$$\mathcal{G}_{h\beta}^{(i_1,i_2)}(T_1,T_2) = h(T_1,T_2)\mathcal{G}_\beta^{(i_1,i_2)}(T_1,T_2).$$

Proof. First of all, observe that by Lemma 5.1.7 and Lemma 5.2.1 we have that $\tilde{h}_\beta$ corresponds to a measure supported on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. Then by Lemma5.1.6 we obtain

$$\Gamma_{\tilde{h}_\beta}^{(i_1-1,-i_2)}(k-1,-j) = D_1^{k-1}D_2^{-j}\tilde{h}_\beta|_{(0,0)} = \left(1-\frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)\Omega_\mathfrak{p}^{1-k}\delta_{k,j}(\beta).$$

On the other hand, Lemma 5.1.4 shows that there exists a power series $\tilde{h}_\beta^{(i_1-1,-i_2)} \in \hat{\mathcal{R}}_\infty[[T_1,T_2]]$ such that for all $s_1, s_2 \in \mathbb{Z}_p$,

$$\Gamma_{\tilde{h}_\beta}^{(i_1-1,-i_2)}(s_1,s_2) = \tilde{h}_\beta^{(i_1-1,-i_2)}(u^{s_1}-1, u^{s_2-1}).$$

We can then define $\mathcal{G}_\beta^{(i_1,i_2)} \in \hat{\mathcal{R}}_\infty[[T_1,T_2]]$ to be

$$\mathcal{G}_\beta^{(i_1,i_2)}(T_1,T_2) = \tilde{h}_\beta^{(i_1-1,-i_2)}(u^{-1}(1+T_1)-1, (1+T_2)^{-1}-1).$$

Observe that then we have

$$\begin{aligned}\mathcal{G}_\beta^{(i_1,i_2)}(u^k-1,u^j-1) &= \tilde{h}_\beta^{(i_1-1,-i_2)}(u^{k-1}-1,u^{-j}-1) = \\ &= \Gamma_{\tilde{h}_\beta}^{(i_1-1,-i_2)}(k-1,-j) = \\ &= \left(1-\frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)\Omega_{\mathfrak{p}}^{1-k}\delta_{k,j}(\beta).\end{aligned}$$

Let now $h \in \Lambda$ and applying equation (4.3) we deduce

$$\begin{aligned}\mathcal{G}_{h(T_1,T_2)\beta}^{(i_1,i_2)}(u^k-1,u^j-1) &= \left(1-\frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)\Omega_{\mathfrak{p}}^{1-k}\delta_{k,j}(h(T_1,T_2)\beta) = \\ &= h(u^k-1,u^j-1)\left(1-\frac{\psi(\mathfrak{p})^{k-j}}{N\mathfrak{p}^{1-j}}\right)\Omega_{\mathfrak{p}}^{1-k}\delta_{k,j}(\beta) = \\ &= h(u^k-1,u^j-1)\mathcal{G}_\beta^{(i_1,i_2)}(u^k-1,u^j-1).\end{aligned}$$

By uniqueness of the power series $\mathcal{G}_\beta^{(i_1,i_2)}(T_1,T_2)$ we conclude

$$\mathcal{G}_{h(T_1,T_2)\beta}^{(i_1,i_2)}(T_1,T_2) = h(T_1,T_2)\mathcal{G}_\beta^{(i_1,i_2)}(T_1,T_2).$$

□

5.3 Interpolation of L-values

For $k > -j \geq 0$ we introduce for simplicity the following notation

$$L_\infty(\overline{\psi}^{k+j},k) = \left(1-\frac{\psi(\mathfrak{p})^{k+j}}{N\mathfrak{p}^{j+1}}\right)\left(1-\frac{\overline{\psi}(\overline{\mathfrak{p}})^{k+j}}{N\overline{\mathfrak{p}}^{k}}\right)\left(\frac{2\pi}{\sqrt{d_K}}\right)\Omega_\infty^{-(k+j)}L(\overline{\psi}^{k+j},k)$$

Let $x \in \mathbb{Z}_p^\times$ and consider $\omega : \mathbb{Z}_p^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ the Teichmuller character, then we have $x = \omega(x)\langle x\rangle$. Recall Definition 5.5 where for u topological generator of $1+p\mathbb{Z}_p$, then we can define a homorphism $l : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ such that

$$\langle x\rangle = u^{l(x)}$$

for every $x \in \mathbb{Z}_p^\times$.

Definition 5.6. *Let $\mu \in \mathcal{S}$ and i_1, i_2 integers modulo $p-1$, then we define*

$$h_\mu^{(i_1,i_2)}(T_1,T_2) = \sum_{\mathfrak{a}\in I}\mu(\mathfrak{a})\left(N\mathfrak{a} - \omega^{i_1}(\psi(\mathfrak{a}))\omega^{i_2}(\overline{\psi}(\mathfrak{a}))(1+T_1)^{l(\psi(\mathfrak{a}))}(1+T_2)^{l(\overline{\psi}(\mathfrak{a}))}\right).$$

Observe that with the previous definition we have for all $(k_1, k_2) \equiv (i_1, i_2) \mod (p-1)$

$$h_\mu^{(i_1,i_2)}(u^{k_1}-1, u^{k_2}-1) = \sum_{\mathfrak{a}\in I} \mu(\mathfrak{a}) \left(N\mathfrak{a} - \omega^{i_1}(\psi(\mathfrak{a}))\omega^{i_2}(\overline{\psi}(\mathfrak{a}))u^{k_1 l(\psi(\mathfrak{a}))}u^{k_2 l(\overline{\psi}(\mathfrak{a}))}\right) =$$
$$= \sum_{\mathfrak{a}\in I} \left(N\mathfrak{a} - \psi(\mathfrak{a})\overline{\psi}(\mathfrak{a})\right)$$

Lemma 5.3.1. *Let $H^{(i_1,i_2)}$ to be the Λ-module generated by $h_\mu^{(i_1,i_2)}(T_1,T_2)$ for all $\mu \in \mathcal{S}$. Then we have*

(i) $H^{(0,0)} = (T_1, T_2)\Lambda$,

(ii) $H^{(1,1)} = (T_1 + 1 - u, T_2 + 1 - u)\Lambda$,

(iii) $H^{(i_1,i_2)} = \Lambda$ *if* $(i_1, i_2) \not\equiv (0,0)$ *and* $(i_1, i_2) \not\equiv (1,1) \mod (p-1)$.

Proof. Let a be an integer which has order $(p-1)$ in the group $(\mathbb{Z}/p^2\mathbb{Z})^\times$. Suppose firstly that $i_1 \not\equiv i_1 \mod (p-1)$. We want to construct an element $\mu_1 \in \mathcal{S}$ such that $h_{\mu_1}^{(i_1,i_2)}$ is a unit in Λ. Let $\alpha_1, \alpha_2 \in \mathcal{O}$ satisfying the following conditions

$$\alpha_1 \equiv 1 \mod \mathfrak{f}\mathfrak{p}, \qquad \alpha_1 \equiv a \mod \overline{\mathfrak{p}},$$
$$\alpha_2 \equiv 1 \mod \mathfrak{f}\overline{\mathfrak{p}}, \qquad \alpha_2 \equiv a \mod \mathfrak{p}$$

with α_1, α_2 coprime with each element of S. Then we have that $\mathfrak{a}_1 = (\alpha_1)$, $\mathfrak{a}_2 = (\alpha_2)$ belong to I. We define $\mu_1 : I \to \mathbb{Z}$ by

$$\mu_1(\mathfrak{a}_1) = N\mathfrak{a}_2 - 1,$$
$$\mu_1(\mathfrak{a}_2) = 1 - N\mathfrak{a}_1,$$
$$\mu_1(\mathfrak{a}) = 0, \quad \text{for all } \mathfrak{a} \neq \mathfrak{a}_1, \mathfrak{a}_2.$$

Then observe we have

$$h_{\mu_1}^{(i_1,i_2)}(0,0) = \sum_{\mathfrak{a}\in I} \mu_1(\mathfrak{a}) \left(N\mathfrak{a} - \omega^{i_1}(\psi(\mathfrak{a}))\omega^{i_2}(\overline{\psi}(\mathfrak{a}))\right) =$$
$$= (N\mathfrak{a}_2 - 1)\left(N\mathfrak{a}_1 - \omega^{i_1}(\psi(\mathfrak{a}_1))\omega^{i_2}(\overline{\psi}(\mathfrak{a}_1))\right) +$$
$$+ (1 - N\mathfrak{a}_1)\left(N\mathfrak{a}_2 - \omega^{i_1}(\psi(\mathfrak{a}_2))\omega^{i_2}(\overline{\psi}(\mathfrak{a}_2))\right)$$

Recall that by definition of the Hecke character ψ we have a multiplicative map from $\{\beta \in \mathcal{O} : (\beta\mathcal{O}, \mathfrak{f}) = 1\}$ to $\mathcal{O}^\times$ defined by $\varepsilon(\beta) = \psi(\beta\mathcal{O})/\beta$. By definition of the conductor, ε factors through $(\mathcal{O}/\mathfrak{f})^\times$. We then conclude $\psi(\mathfrak{a}_1) = \alpha_1$, $\psi(\mathfrak{a}_2) = \alpha_2$ since $\varepsilon(\alpha_1) = \varepsilon(\alpha_2) = \varepsilon(1)$. In particular, we then deduce

$$\omega^{i_1}(\psi(\mathfrak{a}_1))\omega^{i_2}(\overline{\psi}(\mathfrak{a}_1)) = \omega^{i_1}(\alpha_1)\omega^{i_2}(\overline{\alpha_1}) = \omega^{i_2}(\overline{\alpha_1})$$
$$\omega^{i_1}(\psi(\mathfrak{a}_2))\omega^{i_2}(\overline{\psi}(\mathfrak{a}_2)) = \omega^{i_1}(\alpha_2)\omega^{i_2}(\overline{\alpha_2}) = \omega^{i_1}(\alpha_2).$$

Considering $h_{\mu_1}^{(i_1,i_2)}(0,0)$ modulo $\mathfrak{p}$ we get

$$h_{\mu_1}^{(i_1,i_2)}(0,0) = (N\mathfrak{a}_2 - 1)\left(N\mathfrak{a}_1 - \omega^{i_2}(\overline{\alpha}_1)\right) + (1 - N\mathfrak{a}_1)\left(N\mathfrak{a}_2 - \omega^{i_1}(\alpha_2)\right) \equiv$$
$$\equiv (a-1)\left(a - a^{i_2}\right) + (1-a)\left(a - a^{i_1}\right) \equiv$$
$$\equiv (a-1)(a^{i_1} - a^{i_2}) \mod \mathfrak{p}.$$

Since $i_1 \neq i_2$ we conclude $h_{\mu_1}^{(i_1,i_2)}(0,0)$ is a unit. Hence in this case $H^{(i_1,i_2)} = \Lambda$.

Consider now the case $i_1 \equiv i_2$ and $i_1 \not\equiv 0, 1 \mod \mathfrak{p}$. Let $\alpha_3, \alpha_4 \in \mathcal{O}$ satisfying the following conditions

$$\alpha_3 \equiv 1 \mod \mathfrak{f}\mathfrak{p}, \qquad \alpha_3 \equiv a \mod \overline{\mathfrak{p}},$$
$$\alpha_4 \equiv 1 \mod \mathfrak{f}\overline{\mathfrak{p}}, \qquad \alpha_4 \equiv -1 \mod \mathfrak{p}$$

with α_3, α_4 coprime with each element of S. Then we have that $\mathfrak{a}_3 = (\alpha_3)$, $\mathfrak{a}_4 = (\alpha_4)$ belong to I. We define $\mu_2 : I \to \mathbb{Z}$ by

$$\mu_2(\mathfrak{a}_3) = N\mathfrak{a}_4 - 1,$$
$$\mu_2(\mathfrak{a}_4) = 1 - N\mathfrak{a}_3,$$
$$\mu_2(\mathfrak{a}) = 0, \quad \text{for all } \mathfrak{a} \neq \mathfrak{a}_3, \mathfrak{a}_4.$$

By the analogous argument of the previous case, we get

$$\begin{aligned} h_{\mu_2}^{(i_1,i_1)}(0,0) =& (N\mathfrak{a}_4 - 1)\left(N\mathfrak{a}_3 - \omega^{i_1}(\psi(\mathfrak{a}_3)\overline{\psi}(\mathfrak{a}_3))\right) + \\ &+ (1 - N\mathfrak{a}_3)\left(N\mathfrak{a}_4 - \omega^{i_1}(\psi(\mathfrak{a}_4)\overline{\psi}(\mathfrak{a}_4))\right) = \\ =& (N\mathfrak{a}_4 - 1)\left(N\mathfrak{a}_3 - \omega^{i_1}(\overline{\alpha_3})\right) + (1 - N\mathfrak{a}_3)\left(N\mathfrak{a}_4 - \omega^{i_1}(\alpha_4)\right). \end{aligned}$$

Considering $h_{\mu_2}^{(i_1,i_1)}(0,0)$ modulo $\mathfrak{p}$ we get

$$h_{\mu_2}^{(i_1,i_1)}(0,0) \equiv 2(a^{i_1} - 1) \mod \mathfrak{p} \quad \text{if } i_1 \text{ is even},$$
$$h_{\mu_2}^{(i_1,i_1)}(0,0) \equiv 2(a^{i_1} - a) \mod \mathfrak{p} \quad \text{if } i_1 \text{ is odd}.$$

Since $i_1 \not\equiv 1, 0$ we conclude $h_{\mu_2}^{(i_1,i_1)}$ is a unit in Λ and then $H^{(i_1,i_1)} = \Lambda$.

It remains to study the cases $(i_1, i_2) \equiv (0,0)$ or $(1,1) \mod (p-1)$. Observe firstly that for all $\mu \in \mathcal{S}$,

$$h_{\mu}^{(0,0)}(0,0) = \sum_{\mathfrak{a} \in I} \mu(\mathfrak{a})(N\mathfrak{a} - 1) = 0,$$
$$h_{\mu}^{(1,1)}(u-1, u-1) = \sum_{\mathfrak{a} \in I} \mu(\mathfrak{a})(N\mathfrak{a} - \psi(\mathfrak{a})\overline{\psi}(\mathfrak{a})) = 0.$$

We then deduce $h_{\mu}^{(0,0)}(T_1, T_2) \in (T_1, T_2)\Lambda$ and $h_{\mu}^{(1,1)}(T_1, T_2) \in (T_1 - u + 1, T_2 - u + 1)\Lambda$. Thus, to prove the lemma it will suffice to show that $H^{(0,0)}$ contains T_1 and T_2 and $H^{(1,1)}$ contains $T_1 - u + 1$ and $T_2 - u + 1$. Observe that it is enough to produce elements that are congruent to the claimed generators modulo $(\mathfrak{p}^2, (1+T_1)^p - 1, (1+T_2)^p - 1)$.

Let $\alpha_5, \alpha_6 \in \mathcal{O}$ satisfying the following conditions

$$\alpha_5 \equiv 1 \mod \mathfrak{f}\overline{\mathfrak{p}}^2, \qquad \alpha_5 \equiv u \mod \mathfrak{p}^2,$$
$$\alpha_6 \equiv 1 \mod \mathfrak{f}\overline{\mathfrak{p}}^2, \qquad \alpha_6 \equiv au \mod \mathfrak{p}^2$$

with α_5, α_6 coprime with each element of S. Then we have that $\mathfrak{a}_5 = (\alpha_5)$, $\mathfrak{a}_6 = (\alpha_6)$ belong to I. We define $\mu_3 : I \to \mathbb{Z}$ by

$$\mu_3(\mathfrak{a}_5) = N\mathfrak{a}_6 - 1,$$
$$\mu_3(\mathfrak{a}_6) = 1 - N\mathfrak{a}_5,$$
$$\mu_3(\mathfrak{a}) = 0, \quad \text{for all } \mathfrak{a} \neq \mathfrak{a}_5, \mathfrak{a}_6.$$

Using the same arguments as before we get

$$\begin{aligned} h_{\mu_3}^{(0,0)}(T_1,T_2) =&(N\mathfrak{a}_6-1)\left(N\mathfrak{a}_5-(1+T_1)^{l(\alpha_5)}(1+T_2)^{l(\overline{\alpha}_5)}\right)+\\ &+(1-N\mathfrak{a}_5)\left(N\mathfrak{a}_6-(1+T_1)^{l(\alpha_6)}(1+T_2)^{l(\overline{\alpha}_6)}\right) \end{aligned}$$

Considering $h_{\mu_3}^{(0,0)}$ modulo $((1+T_1)^p-1,(1+T_2)^p-1)$ we obtain

$$h_{\mu_3}^{(0,0)}(T_1,T_2)\equiv(1-a)uT_1 \mod ((1+T_1)^p-1,(1+T_2)^p-1).$$

and so $H^{(0,0)}$ contains an element congruent to T_1 modulo $((1+T_1)^p-1,(1+T_2)^p-1)$. The construction of an element congruent to T_2 is completely symmetrical.

Let $\alpha_7,\alpha_8\in\mathcal{O}$ satisfying the following conditions

$$\begin{aligned} \alpha_7&\equiv 1 \mod \mathfrak{f}\mathfrak{p}^2, & \alpha_7&\equiv u \mod \overline{\mathfrak{p}}^2,\\ \alpha_8&\equiv 1 \mod \mathfrak{f}\mathfrak{p}^2, & \alpha_8&\equiv a \mod \overline{\mathfrak{p}}^2 \end{aligned}$$

with α_7,α_8 coprime with each element of S. Then we have that $\mathfrak{a}_7=(\alpha_7)$, $\mathfrak{a}_8=(\alpha_8)$ belong to I. We define $\mu_4: I\to\mathbb{Z}$ by

$$\begin{aligned} &\mu_4(\mathfrak{a}_7)=N\mathfrak{a}_8-1,\\ &\mu_4(\mathfrak{a}_8)=1-N\mathfrak{a}_7,\\ &\mu_4(\mathfrak{a})=0, \quad \text{for all } \mathfrak{a}\neq\mathfrak{a}_7,\mathfrak{a}_8. \end{aligned}$$

Using the same arguments as before we get

$$\begin{aligned} h_{\mu_4}^{(1,1)}(T_1,T_2) =&(N\mathfrak{a}_8-1)\left(N\mathfrak{a}_7-\omega(N\mathfrak{a}_7)(1+T_1)^{l(\alpha_7)}(1+T_2)^{l(\overline{\alpha}_7)}\right)+\\ &+(1-N\mathfrak{a}_7)\left(N\mathfrak{a}_8-\omega(N\mathfrak{a}_8)(1+T_1)^{l(\alpha_8)}(1+T_2)^{l(\overline{\alpha}_8)}\right) \end{aligned}$$

Considering $h_{\mu_4}^{(1,1)}$ modulo $(\mathfrak{p}^2,(1+T_1)^p-1,(1+T_2)^p-1)$ we obtain

$$h_{\mu_4}^{(1,1)}(T_1,T_2)\equiv(1-a)(T_2+1-u) \mod ((1+T_1)^p-1,(1+T_2)^p-1).$$

and so $H^{(1,1)}$ contains an element congruent to T_2-u+1 modulo $(\mathfrak{p}^2,(1+T_1)^p-1,(1+T_2)^p-1)$. The construction of an element congruent to T_1-u+1 is completely symmetrical. □

Theorem 5.3. *Let i_1,i_2 be integers modulo $p-1$. Then there is a power series $\mathcal{G}^{(i_1,i_2)}(T_1,T_2)\in\hat{\mathcal{R}}_\infty[[T_1,T_2]]$ such that, for all integers $k_1>-k_2\geq 0$ and $(k_1,k_2)\equiv(i_1,i_2) \mod (p-1)$,*

$$\mathcal{G}^{(i_1,i_2)}(u^{k_1}-1,u^{k_2}-1)=(k_1-1)!\Omega_{\mathfrak{p}}^{k_2-k_1}L_\infty(\overline{\psi}^{k_1-k_2},k_1)$$

Proof. Let $\mu\in\mathcal{S}$, consider the elliptic unit $\langle e(\mu)\rangle\in C_\infty$ defined in Corollary 3.7.3 then by Theorem 5.2 and Theorem 4.6 we have

$$\begin{aligned} \mathcal{G}_{\langle e(\mu)\rangle}^{(i_1,i_2)}(u^{k_1}-1,u^{k_2}-1)=&\left(1-\frac{\psi(\mathfrak{p})^{k_1-k_2}}{N\mathfrak{p}^{1-k_2}}\right)\Omega_{\mathfrak{p}}^{1-k_1}\delta_{k_1,k_2}(\langle e(\mu)\rangle)=\\ =&\Omega_{\mathfrak{p}}^{1+k_2-k_1}12(-1)^{k_1+1-k_2}(k_1-1)!f^{k_1}\\ &h_\mu^{i_1,i_2}(u^{k_1}-1,u^{k_2}-1)L_\infty(\overline{\psi}^{k_1-k_2},k_1). \end{aligned}$$

Consider the power series $g(T_1, T_2) \in \mathbb{Z}_p[[T_1, T_2]]$ defined by

$$g(T_1, T_2) = 12(-1)^{1+i_1-i_2}\omega^{i_1}(f)(1+T_1)^{l(f)}.$$

In particular we have $g(u^{k_1} - 1, u^{k_2} - 1) = 12(-1)^{1+k_1-k_2} f^{k_1}$ whenever $(k_1, k_2) \equiv (i_1, i_2) \mod (p-1)$. Observe that $g(T_1, T_2)$ is a unit in Λ since p is coprime with $12f$. Consider then $h(T_1, T_2) = g(T_1, T_2)h_\mu(T_1, T_2) \in \Lambda$, we can rewrite the previous equation as

$$\mathcal{G}^{(i_1,i_2)}_{\langle e(\mu)\rangle}(u^{k_1} - 1, u^{k_2} - 1) = h(u^{k_1} - 1, u^{k_2} - 1)\Omega_{\mathfrak{p}}^{1+k_2-k_1}(k_1 - 1)!L_\infty(\overline{\psi}^{k_1-k_2}, k_1).$$

If $(i_1, i_2) \not\equiv (0, 0), (1, 1) \mod (p-1)$ then by the previous lemma there exists $\mu \in \mathcal{S}$ such that h_μ is invertible. In this case, we consider $\beta \in U_\infty$ to be the unit given by $h^{-1}(T_1, T_2)\langle e(\mu)\rangle$, and then by Theorem 5.2 we have that the value of the power series $\mathcal{G}^{(i_1,i_2)}_\beta$ at $(u^{k_1} - 1, u^{k_2} - 1)$ is

$$\begin{aligned}\mathcal{G}^{(i_1,i_2)}_\beta(u^{k_1} - 1, u^{k_2} - 1) &= h^{-1}(u^{k_1} - 1, u^{k_2} - 1)\mathcal{G}^{(i_1,i_2)}_{\langle e(\mu)\rangle}(u^{k_1} - 1, u^{k_2} - 1) = \\ &= \Omega_{\mathfrak{p}}^{1+k_2-k_1}(k_1 - 1)!L_\infty(\overline{\psi}^{k_1-k_2}, k_1).\end{aligned}$$

It remains to study the cases $(i_1, i_2) \equiv (0, 0)$ and $(i_1, i_2) \equiv (1, 1) \mod (p-1)$.

Suppose $(i_1, i_2) \equiv (0, 0) \mod (p-1)$. By the previous lemma we have $H^{(0,0)} = (T_1, T_2)\Lambda$. Let e_0 be the unit in D corresponding to the power series $h_{e_0} = T_2 \in H^{(0,0)}$. We then have $\mathcal{G}^{(0,0)}_{e_0}(u^{k_1} - 1, 0) = 0$ for all $k_1 \equiv 0 \mod (p-1)$. In particular

$$\mathcal{G}^{(0,0)}_{e_0}(T_1, T_2) = \Omega_{\mathfrak{p}} T_2 \mathcal{G}^{(0,0)}(T_1, T_2)$$

for some power series $\mathcal{G}^{(0,0)}(T_1, T_2)$. From the previous equation $\mathcal{G}^{(0,0)}(T_1, T_2)$ has the desired properties.

Suppose $(i_1, i_2) \equiv (1, 1) \mod (p-1)$. Let e_1 be the unit in D corresponding to the power series $h_{e_1} = T_1 + 1 - u \in H^{(1,1)}$. We then have $\mathcal{G}^{(0,0)}_{e_0}(u - 1, u^{k_2} - 1) = 0$ for all $k_2 \equiv 1 \mod (p-1)$. In particular

$$\mathcal{G}^{(1,1)}_{e_1}(T_1, T_2) = \Omega_{\mathfrak{p}}(T_1 + 1 - u)\mathcal{G}^{(1,1)}(T_1, T_2)$$

for some power series $\mathcal{G}^{(1,1)}(T_1, T_2)$. From the previous equation $\mathcal{G}^{(1,1)}(T_1, T_2)$ has the desired properties. □

CHAPTER 6

Two variable theta function

6.1 Two variable Theta function

K. Bannai and S. Kobayashi [BK10] studied a particular type of two variable Theta function that is the generating function for Eisenstein-Kronecker numbers. Through Mumford's theory, they reproved the algebraicity of these numbers and they constructed a p-adic measure using the formal power series associated.

6.1.1 Definition and Laurent expansion

Recall that we previously defined the functions for a lattice $L = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$

$$\sigma(z, L) = z \prod_{\omega \in L - \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2\right)$$
$$s_2(L) = \lim_{s \to 0^+} \sum_{\omega \in L - \{0\}} \omega^{-2} |\omega|^{-2s}.$$

Definition 6.1. *We define the Kronecker theta function for L to be*

$$\Theta(z, w, L) = \exp\left(-s_2 zw\right) \frac{\sigma(z + w, L)}{\sigma(z, L)\sigma(w, L)}.$$

Prop. 6.1.1 (Transformation formula). *For any $\gamma_1, \gamma_2 \in L$ we have*

$$\Theta(z + \gamma_1, w + \gamma_2, L) = \exp\left(\frac{\gamma_1 \overline{\gamma_2}}{A(L)}\right) \exp\left(\frac{z\overline{\gamma_2} + w\overline{\gamma_1}}{A(L)}\right) \Theta(z, w, L). \tag{6.1}$$

For any $c \in \mathbb{C}$ we have

$$\Theta(cz, cw, cL) = \frac{1}{c}\Theta(z, w, L) \tag{6.2}$$

Lemma 6.1.1. *Let $f(z, w) = \exp(z\overline{w}/A)H_1(z, w, 1)$. Then this function satisfies the following properties*

(i) *$f(z, w)$ satisfies the transformation formula* (6.1)*,*

(ii) *$f(z, w)$ is a meromorphic function in z and w, holomorphic except simple poles when $z \in L$ or $w \in L$,*

(iii) *the residue of $f(z,w)$ at $z=0$ and $w=0$ is equal to one.*

Lemma 6.1.2. *Any holomorphic function $f(z,w)$ on $\mathbb{C}\times\mathbb{C}$ satisfying the transformation formula*

$$f(z+u,w+v)=\exp\left(\frac{u\overline{v}}{A}\right)\exp\left(\frac{z\overline{v}+w\overline{u}}{A}\right)$$

for any $u,v\in L$ is identically equal to zero.

Theorem 6.1 (Kronecker). *The Kronecker theta function is related to the Eisenstein-Kronecker-Lerch series as follows*

$$\Theta(z,w)=\exp\left(\frac{z\overline{w}}{A}\right)H_1(z,w,1).$$

In particular, the theta function $\Theta(z,w)$ is holomorphic on $\mathbb{C}\times\mathbb{C}$ except for simple poles corresponding to $z\in\Gamma$ or $w\in\Gamma$. In particular $\exp(z\overline{w}/A)H_1(z,w,1)$ has residue 1 along $z=0$ and $w=0$.

Proof. See [BK10].1.13. □

Definition 6.2. *For any $z_0,w_0\in\mathbb{C}$ we define the translated Kronecker theta function by*

$$\Theta_{z_0,w_0}(z,w,L)=\exp\left(-\frac{z_0\overline{w}_0}{A(L)}\right)\exp\left(-\frac{z\overline{w}_0+w\overline{z}_0}{A(L)}\right)\Theta(z+z_0,w+w_0,L).$$

Lemma 6.1.3. *For any $z_0,w_0\in\mathbb{C}$ and $\gamma,\gamma'\in L$ we have that the translated Kronecker theta function satisfies the following relation*

$$\Theta_{z_0+\gamma,w_0+\gamma'}(z,w,L)=\langle w_0,\gamma\rangle_L\Theta_{z_0,w_0}(z,w,L). \tag{6.3}$$

For any $c\in\mathbb{C}$ we have

$$\Theta_{cz_0,cw_0}(cz,cw,cL)=\frac{1}{c}\Theta_{z_0,w_0}(z,w,L). \tag{6.4}$$

Proof. It follows by a straightforward computation. By definition and the transformation formula (6.1) we have that the left-hand side is

$$\begin{aligned}\Theta_{z_0+\gamma,w_0+\gamma'}(z,w)=&\exp\left(-\frac{(z_0+\gamma)\overline{(w_0+\gamma')}}{A}\right)\exp\left(-\frac{z\overline{(w_0+\gamma')}+w\overline{(z_0+\gamma)}}{A}\right)\\&\Theta(z+z_0+\gamma,w+w_0+\gamma')=\\=&\exp\left(-\frac{(z_0+\gamma)\overline{(w_0+\gamma')}}{A}\right)\exp\left(-\frac{z\overline{(w_0+\gamma')}+w\overline{(z_0+\gamma)}}{A}\right)\\&\exp\left(\frac{\gamma\overline{\gamma}'}{A}\right)\exp\left(\frac{(z+z_0)\overline{\gamma}'+(w+w_0)\overline{\gamma}}{A}\right)\Theta(z+z_0,w+w_0).\end{aligned}$$

The right side is equal to

$$\langle w_0,\gamma\rangle\Theta_{z_0,w_0}(z,w)=\exp\left(\frac{w_0\overline{\gamma}-\gamma\overline{w}_0}{A}\right)\exp\left(-\frac{z_0\overline{w}_0}{A}\right)\exp\left(-\frac{z\overline{w}_0+w\overline{z}_0}{A}\right)\Theta(z+z_0,w+w_0).$$

Comparing the exponential factor we conclude the desired equation.

For the homothety relation observe that we have

$$\Theta_{cz_0,cw_0}(cz,cw,cL)=\exp\left(-\frac{cz_0\overline{cw}_0}{A(cL)}\right)\exp\left(-\frac{cz\overline{cw}_0+cw\overline{cz}_0}{A(cL)}\right)\Theta(cz+cz_0,cw+cw_0,cL)$$

and using the fact that $A(cL) = NcA(L)$ and the homothety relation (6.2) we conclude

$$\Theta_{cz_0,cw_0}(cz, cw, cL) = \frac{1}{c}\Theta_{z_0,w_0}(z, w, L).$$

□

Lemma 6.1.4. *For any $\gamma \in L$, we have*

$$\lim_{z\to -u+\gamma}(z+u-\gamma)\Theta_{u,v}(z,w) = \langle v, \gamma\rangle \exp\left(\frac{(\overline{\gamma}-\overline{u})w}{A}\right)$$
$$\lim_{w\to -v+\gamma}(w+v-\gamma)\Theta_{u,v}(z,w) = \langle u, \gamma - v\rangle \exp\left(\frac{z(\overline{\gamma}-\overline{v})}{A}\right)$$

Proof. This follows from direct calculations applying Kronecker's theorem. In fact we have

$$\Theta_{u,v}(z,w) = \exp\left(-\frac{u\overline{v}}{A}\right)\exp\left(-\frac{z\overline{v}+w\overline{u}}{A}\right)\Theta(z+u, w+v).$$

By Kronecker's theorem we know that $\Theta(z, w)$ has a simple pole at $z \in \Gamma$, $w \in \Gamma$ with residue 1 along $z = 0$ and $w = 0$. To calculate the residues we use the transformation formula (6.1)

$$\Theta(z+\gamma, w) = \exp\left(\frac{w\overline{\gamma}}{A}\right)\Theta(z, w)$$

deducing that the residue of $\Theta(z, w)$ at $z = \gamma$ is $\exp(w\overline{\gamma}/A)$ and then

$$\lim_{z\to -u+\gamma}(z+u-\gamma)\exp\left(-\frac{(z+u)\overline{v}+w\overline{u}}{A}\right)\Theta(z+u, w+v) =$$
$$= \exp\left(-\frac{\gamma\overline{v}+w\overline{u}}{A}\right)\exp\left(\frac{(w+v)\overline{\gamma}}{A}\right) = \langle v,\gamma\rangle \exp\left(\frac{(\overline{\gamma}-\overline{u})w}{A}\right).$$

The other case is analogous. □

In the same fashion as the previous definition of the theta function, the two variable form satisfies a distribution relation.

Prop. 6.1.2 (Distribution relation). *Let $\mathfrak{a}, \mathfrak{b}$ be integral ideals of $\mathcal{O}_K$ such that $(\mathfrak{ab}, \overline{\mathfrak{b}}) = 1$. Let $\epsilon \in \mathcal{O}_K$ be such that $\epsilon \equiv 1 \mod \mathfrak{ab}$ and $\epsilon \equiv 0 \mod \overline{\mathfrak{b}}$. Then*

$$\sum_{\substack{\alpha\in\mathfrak{a}^{-1}L/L\\ \beta\in\mathfrak{b}^{-1}L/L}} \langle\epsilon\alpha, w_0\rangle_L \Theta_{z_0+\epsilon\alpha, w_0+\epsilon\beta}(z, w, L) = N(\mathfrak{ab})\Theta_{N\mathfrak{a}z_0, N\mathfrak{b}w_0}(N\mathfrak{a}z, N\mathfrak{b}w, \overline{\mathfrak{ab}}L) \qquad (6.5)$$

Proof. First of all observe that the quantity $\langle\epsilon\alpha, w_0\rangle_L\Theta_{z_0+\epsilon\alpha,w_0+\epsilon\beta}(z, w, L)$ do not depend on the choice of representatives of α and β. Indeed, let $\alpha' = \alpha + \gamma_1$ and $\beta' = \beta + \gamma_2$ with $\gamma_1, \gamma_2 \in L$, then by the previous lemma we get

$$\Theta_{z_0+\epsilon\alpha', w_0+\epsilon\beta'}(z, w) = \langle w_0+\epsilon\beta, \epsilon\gamma_1\rangle\Theta_{z_0+\epsilon\alpha,w_0+\epsilon\beta}(z, w) =$$
$$= \langle w_0, \epsilon\gamma_1\rangle\Theta_{z_0+\epsilon\alpha,w_0+\epsilon\beta}(z, w)$$

where we have used the fact that $\langle \epsilon\beta, \epsilon\gamma_1\rangle = \langle \epsilon\overline{\epsilon}\beta, \gamma_1\rangle$ by Lemma 3.2.1 and $\overline{\epsilon}\beta \in L$ since $\overline{\epsilon} \in \mathfrak{b}$. From this we deduce

$$\langle \epsilon\alpha, w_0\rangle_L \Theta_{z_0+\epsilon\alpha,w_0}(z,w,L) = \langle \epsilon\alpha', w_0\rangle_L \Theta_{z_0+\epsilon\alpha',w_0}(z,w,L).$$

We will now consider the case $z_0 = 0$, $w_0 = 0$, see [BK10].1.16 for the translated case. We show that both sides have the same transformation formula with respect to $\overline{\mathfrak{a}\mathfrak{b}}L$. For $u, v \in \mathfrak{a}\mathfrak{b}L$, we have by equation (6.1)

$$\Theta(N\mathfrak{a}(z+u), N\mathfrak{b}(w+v), \overline{\mathfrak{a}\mathfrak{b}}L) = \exp\left(\frac{N\mathfrak{a}u\overline{N\mathfrak{b}v}}{A(\overline{\mathfrak{a}\mathfrak{b}}L)}\right)\exp\left(\frac{z\overline{N\mathfrak{b}v} + w\overline{N\mathfrak{a}u}}{A(\overline{\mathfrak{a}\mathfrak{b}}L)}\right)\Theta(N\mathfrak{a}z, N\mathfrak{b}w, \overline{\mathfrak{a}\mathfrak{b}}L) =$$
$$= \exp\left(\frac{u\overline{v}}{A(L)}\right)\exp\left(\frac{z\overline{v}+w\overline{u}}{A(L)}\right)\Theta(N\mathfrak{a}z, N\mathfrak{b}w, \overline{\mathfrak{a}\mathfrak{b}}L). \quad (6.6)$$

Now observe that by Lemma 3.2.1 we have $\langle \epsilon\alpha, v\rangle_L = \langle \epsilon, \overline{\alpha}v\rangle_L$ and then since $v \in \overline{\mathfrak{a}\mathfrak{b}}L$ and $\alpha \in \alpha^{-1}L/L$ we conclude $\langle \epsilon\alpha, v\rangle_L = 1$. Analogously we have $\langle \epsilon\beta, u\rangle_L = 1$. Therefore, using Lemma 6.1.3, we deduce

$$\Theta_{\epsilon\alpha+u,\epsilon\beta+v}(z,w,L) = \langle \epsilon\beta, u\rangle_L \Theta_{\epsilon\alpha,\epsilon\beta}(z,w,L) = \Theta_{\epsilon\alpha,\epsilon\beta}(z,w,L).$$

Observe that we have

$$\Theta_{\epsilon\alpha,\epsilon\beta}(z+u, w+v, L) = \exp\left(-\frac{\epsilon\alpha\overline{\epsilon\beta}}{A}\right)\exp\left(-\frac{(z+u)\overline{\epsilon\beta} + (w+v)\overline{\epsilon\alpha}}{A}\right)\Theta(z+\epsilon\alpha+u, w+\epsilon\alpha+v, L) =$$
$$\Theta_{\epsilon\alpha+u,\epsilon\beta+v}(z,w,L) = \exp\left(-\frac{(\epsilon\alpha+u)\overline{(\epsilon\beta+v)}}{A}\right)\exp\left(-\frac{z\overline{(\epsilon\beta+v)} + w\overline{(\epsilon\alpha+u)}}{A}\right)$$
$$\Theta(z+\epsilon\alpha+u, w+\epsilon\beta+v)$$

and hence comparing the exponential factor we deduce the following translation relation

$$\Theta_{\epsilon\alpha,\epsilon\beta}(z+u, w+v, L) = \exp\left(\frac{u\overline{v}}{A(L)}\right)\exp\left(\frac{z\overline{v}+w\overline{u}}{A(L)}\right)\Theta_{\epsilon\alpha+u,\epsilon\beta+v}(z,w,L) =$$
$$= \exp\left(\frac{u\overline{v}}{A(L)}\right)\exp\left(\frac{z\overline{v}+w\overline{u}}{A(L)}\right)\Theta_{\epsilon\alpha,\epsilon\beta}(z,w,L). \quad (6.7)$$

From this equation and (6.6) we conclude that both satisfy the same transformation formula with respect to $\overline{\mathfrak{a}\mathfrak{b}}L$.

Next, we show that both sides have the same poles with the same residues. By Kronecker's theorem 6.1 we have that the left hand side of (6.5) has simple poles at most on (z, w) where $z = -\epsilon\alpha_0 + \gamma$ or $w = -\epsilon\beta_0 + \gamma$ for some $\alpha_0 \in \mathfrak{a}^{-1}L$, $\beta_0 \in \mathfrak{b}^{-1}L$ and $\gamma \in L$. By the previous lemma, we have

$$\lim_{z\to -\epsilon\alpha_0+\gamma}(z+\epsilon\alpha_0+\gamma)\sum_{\substack{\alpha\in\mathfrak{a}^{-1}L/L\\ \beta\in\mathfrak{b}^{-1}L/L}}\Theta_{\epsilon\alpha,\epsilon\beta}(z,w,\Gamma) = \sum_{\beta\in\mathfrak{b}^{-1}L/L}\langle \epsilon\beta,\gamma\rangle \exp\left(\frac{(\overline{\gamma}-\overline{\epsilon\alpha_0})w}{A}\right) =$$
$$= \begin{cases} N(\mathfrak{b})\exp\left(\frac{(\overline{\gamma}-\overline{\epsilon\alpha_0})w}{A}\right) & \text{if } \gamma \in \overline{\mathfrak{b}}L \\ 0 & \text{otherwise.}\end{cases}$$

Hence the left hand side has a pole at $z \in (\epsilon\mathfrak{a}^{-1} + \overline{\mathfrak{b}})L = \overline{\mathfrak{b}}\mathfrak{a}^{-1}L$. By Kronecker's theorem 6.1 we have that the righ hand side of equation (6.5) has the same poles for z with the same residues. The case of w is analogous. □

Theorem 6.2. *For any $z_0, w_0 \in \mathbb{C}$ we have that the Laurent expansion of $\Theta_{z_0,w_0}(z,w)$ at the origin is given by*

$$\Theta_{z_0,w_0}(z,w) = \langle w_0, z_0 \rangle \delta_{z_0} z^{-1} + \delta_{w_0} w^{-1} + \sum_{k,j \geq 0} (-1)^{k+j} \frac{E_{j,k+1}(z_0,w_0)}{A^j j!} z^k w^j$$

where $\delta_x = 1$ if $x \in L$ and is zero otherwise.

Proof. Let $\tilde{H}_{j+k}(z,w,k) = \exp(-w\overline{z}/A) H_{j+k}(z,w,k)$. Then by Lemma 3.2.2 we obtain

$$\partial_z \tilde{H}_{j+k}(z,w,k) = -k\tilde{H}_{j+k+1}(z,w,k)$$
$$\partial_w \tilde{H}_{j+k}(z,w,k) = -\frac{1}{A}\tilde{H}_{j+k+1}(z,w,k).$$

Hence when $z_0, w_0 \notin \Gamma$, the coefficient of z and w in the Taylor expansion of $\tilde{H}_1(z+z_0, w+w_0, 1)$ at the origin is given by

$$\sum_{k,j\geq 0} (-1)^{k+j} \frac{\tilde{H}_{k+j+1}(z_0,w_0,k+1)}{j!A^j} z^k w^j.$$

By definition of $\Theta_{(z_0,w_0)}(z,w)$ and Kronecker's theorem we get

$$\Theta_{(z_0,w_0)}(z,w) = \exp\left(\frac{w_0\overline{z_0}}{A}\right) \exp\left(\frac{(z+z_0)\overline{w} + (w+w_0)\overline{z}}{A}\right) \tilde{H}_1(z+z_0, w+w_0, 1).$$

Sine $\Theta_{(z_0,w_0)}(z,w)$ is holomorphic at the origin, the assertion follows form the fact that $E_{j,k}(z_0,w_0) = \exp(w_0\overline{z}_0/A)\tilde{H}_{j+k}(z_0,w_0,k)$. The case when $z_0, w_0 \in L$ follows using similar argument, paying careful attention to the poles. See [BK10] 1.4. □

6.1.2 Tables of values

Using the results of the previous sections, we can now compute the values of the Eisentein numbers $E_{j,k}$ through the expansion of the two variables theta function $\Theta_{(z_0,w_0)}(z,w)$. In the following tables, we show the coefficients of the expansion.

$L = \Omega_\infty(\mathbb{Z} + i\,\mathbb{Z})$ with $\Omega_\infty \approx 1.854074677301371918433850347200 - 1.854074677301371918433850347200i$

$\Theta_{(0,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	0	0	$\frac{1}{15}$	0	0	0	$-\frac{1}{525}$
y	0	0	$\frac{1}{6}$	0	0	0	$-\frac{2}{315}$	0
y^2	0	$\frac{1}{6}$	0	0	0	$-\frac{1}{90}$	0	0
y^3	$\frac{1}{15}$	0	0	0	$-\frac{1}{72}$	0	0	0
y^4	0	0	0	$-\frac{1}{72}$	0	0	0	$\frac{1}{7560}$
y^5	0	0	$-\frac{1}{90}$	0	0	0	$\frac{1}{10800}$	0
y^6	0	$-\frac{1}{315}$	0	0	0	$\frac{1}{10800}$	0	0
y^7	$-\frac{1}{525}$	0	0	0	$\frac{1}{7560}$	0	0	0

$\Theta_{(w_1/2,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	$\frac{1}{2}i$	0	$\frac{1}{40}$	0	$-\frac{1}{80}i$	0	$\frac{1}{9600}$
y	i	0	$\frac{1}{6}$	0	$-\frac{1}{24}i$	0	$\frac{1}{1680}$	0
y^2	0	$\frac{1}{2}$	0	$-\frac{1}{12}i$	0	$\frac{1}{720}$	0	$-\frac{13}{10080}i$
y^3	$\frac{2}{3}$	0	$-\frac{1}{6}i$	0	0	0	$\frac{1}{720}i$	0
y^4	0	$-\frac{1}{3}i$	0	$-\frac{1}{72}$	0	$\frac{1}{720}i$	0	$\frac{1}{20160}$
y^5	$-\frac{2}{5}i$	0	$-\frac{2}{30}$	0	$\frac{1}{360}i$	0	$\frac{1}{10800}$	0
y^6	0	$-\frac{8}{45}$	0	$-\frac{1}{90}i$	0	$\frac{1}{3600}$	0	$-\frac{1}{151200}i$
y^7	$-\frac{8}{35}$	0	$\frac{13}{315}$	0	$\frac{1}{756}$	0	$-\frac{1}{75600}$	0

$L = \Omega_\infty(\mathbb{Z} + \sqrt{-2}\,\mathbb{Z})$ with $\Omega_\infty \approx -0.17382248014992879654865318312 2i$

$\Theta_{(0,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	35	0	-2450	0	68600	0	-2572500
y	35	0	$-\frac{11024}{2}$	0	154350	0	-6174000	0
y^2	0	$-\frac{11024}{2}$	0	$\frac{385874}{2}$	0	-8103375	0	162067500
y^3	-2450	0	$\frac{385874}{2}$	0	$-\frac{67528125}{8}$	0	94539375	0
y^4	0	154350	0	$-\frac{67528125}{8}$	0	$\frac{283618125}{8}$	0	$-\frac{15598996875}{4}$
y^5	68600	0	-8103375	0	$\frac{283618125}{8}$	0	$-\frac{45662518125}{16}$	0
y^6	0	-6174000	0	94539375	0	$-\frac{45662518125}{16}$	0	$\frac{287872396875}{16}$
y^7	-2572500	0	162067500	0	$-\frac{15598996875}{4}$	0	$\frac{287872396875}{16}$	0

$\Theta_{(w_1/2,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	-70	0	1225	0	-85750	0	$\frac{4501875}{2}$
y	-105	0	0	0	-231525	0	4630500	0
y^2	0	$-\frac{11025}{2}$	0	-385875	0	$\frac{8103375}{2}$	0	-202584375
y^3	-7350	0	$-\frac{1157625}{2}$	0	0	0	$-\frac{283618125}{2}$	0
y^4	0	-771750	0	$-\frac{67528125}{8}$	0	$-\frac{-283618125}{4}$	0	$\frac{15598996875}{8}$
y^5	-617400	0	-24310125	0	$-\frac{850854375}{8}$	0	0	0
y^6	0	-43218000	0	-472696875	0	$-\frac{45662518125}{16}$	0	$-\frac{287872396875}{8}$
y^7	-38587500	0	-1458607500	0	$-\frac{46796990625}{4}$	0	$-\frac{863617190625}{16}$	0

$L = \Omega_\infty(\mathbb{Z} + \frac{1+\sqrt{-3}}{2}\,\mathbb{Z})$ with $\Omega_\infty \approx 2.1032731579881813917625286185 8 - 1.2143253239437908059099708448 9i$

$\Theta_{(0,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	0	0	0	0	$\frac{1}{35}$	0	0
y	0	0	0	0	$\frac{1}{10}$	0	0	0
y^2	0	0	0	$\frac{1}{6}$	0	0	0	0
y^3	0	0	$\frac{1}{6}$	0	0	0	0	0
y^4	0	$\frac{1}{10}$	0	0	0	0	0	$-\frac{1}{840}$
y^5	$\frac{1}{35}$	0	0	0	0	0	$-\frac{1}{900}$	0
y^6	0	0	0	0	0	$-\frac{1}{900}$	0	0
y^7	0	0	0	0	$-\frac{1}{840}$	0	0	0

$\Theta_{(w_1/2,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	$\frac{1}{2}$	0	$-\frac{1}{8}$	0	$-\frac{1}{112}$	0	$-\frac{3}{896}$
y	1	0	$-\frac{1}{2}$	0	$-\frac{1}{40}$	0	$-\frac{1}{80}$	0
y^2	0	-1	0	0	0	$-\frac{1}{40}$	0	$-\frac{1}{280}$
y^3	-1	0	$\frac{1}{6}$	0	$-\frac{1}{24}$	0	$\frac{1}{240}$	0
y^4	0	$\frac{1}{2}$	0	$\frac{1}{12}$	0	$\frac{1}{240}$	0	$\frac{1}{3360}$
y^5	$\frac{3}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{120}$	0	0	0
y^6	0	$-\frac{2}{5}$	0	$\frac{1}{30}$	0	$-\frac{1}{900}$	0	$\frac{1}{25200}$
y^7	$-\frac{3}{7}$	0	$\frac{4}{35}$	0	$-\frac{1}{168}$	0	$\frac{1}{12600}$	0

$L = \Omega_\infty(\mathbb{Z} + \frac{1+\sqrt{-7}}{2}\,\mathbb{Z})$ with $\Omega_\infty \approx 0.2495894679627255757672578641846$

$\Theta_{(0,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	-15	0	-525	0	-9450	0	-118125
y	-15	0	-1200	0	-25200	0	-252000	0
y^2	0	-1200	0	-36000	0	-252000	0	-7560000
y^3	-525	0	-36000	0	-180000	0	-7560000	0
y^4	0	-25200	0	-180000	0	-7020000	0	-37800000
y^5	-9450	0	-252000	0	-7020000	0	-18360000	0
y^6	0	-252000	0	-7560000	0	-18360000	0	$-\frac{745200000}{7}$
y^7	-118125	0	-7560000	0	-37800000	0	$-\frac{745200000}{7}$	0

$\Theta_{(w_1/2,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	$\frac{105}{4}+\frac{15\sqrt{-7}}{4}$	0	$\frac{7875}{16}-\frac{1575\sqrt{-7}}{16}$	0	$\frac{259875}{32}+\frac{23625\sqrt{-7}}{32}$	0	$\frac{33901875}{256}-\frac{354375\sqrt{-7}}{256}$
y	$\frac{75}{2}+\frac{15\sqrt{-7}}{2}$	1 0	$1050-450\sqrt{-7}$	0	$\frac{36225}{2}+\frac{7875\sqrt{-7}}{2}$	0	$\frac{1252125}{4}-\frac{23625\sqrt{-7}}{4}$	0
y^2	- 0	$900-900\sqrt{-7}$	0	$15750+11250\sqrt{7}$	0	$\frac{748125}{2}-\frac{23625\sqrt{-7}}{2}$	0	$\frac{23506875}{4}-\frac{1299375\sqrt{-7}}{4}$
y^3	$\frac{525}{2}-\frac{1575\sqrt{-7}}{2}$	0	$-4500+22500\sqrt{-7}$	0	$354375-16875\sqrt{-7}$	0	$\frac{8386875}{2}-\frac{1299375\sqrt{-7}}{2}$	0
y^4	0	$-31500+31500\sqrt{-7}$	0	$528750-33750\sqrt{-7}$	0	$\frac{1535625}{2}-\frac{2413125 sqrt-7}{2}$	0	$\frac{178959375}{4}+\frac{26578125\sqrt{-7}}{4}$
y^5	$-33075+23625\sqrt{-7}$	0	$1228500-94500\sqrt{-7}$	0	$-5484375-2413125\sqrt{-7}$	0	$\frac{50203125}{2}+\frac{12909375\sqrt{-7}}{2}$	0
y^6	0	$2205000-189000\sqrt{-7}$	0	$-19372500-5197500\sqrt{7}$	0	$31843125+12909375\sqrt{-7}$	0	$\frac{285271875}{2}-\frac{75684375\sqrt{-7}}{2}$
y^7	$\frac{3898125}{2}-\frac{354375\sqrt{-7}}{2}$	0	$-46305000-10395000\sqrt{-7}$	0	$93318750+53156250\sqrt{7}$	0	$\frac{1251703125}{7}+75684375\sqrt{-7}$	0

$L=\Omega_\infty(\mathbb{Z}+\frac{1+\sqrt{-11}}{2}\mathbb{Z})$ with $\Omega_\infty \approx 0.157988062436041406847226542091$

$\Theta_{(0,0)}(x,y)$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	-56	0	$-\frac{17248}{5}$	0	$-\frac{664048}{5}$	0	$-\frac{127497216}{25}$
y	-56	0	-7056	0	-271656	0	$-\frac{47811456}{5}$	0
y^2	0	-7056	0	-321048	0	$-\frac{45638208}{5}$	0	$-\frac{2008081152}{5}$
y^3	$-\frac{17248}{5}$	0	-321048	0	-6914880	0	$-\frac{1490848128}{5}$	0
y^4	0	-271656	0	-6914880	0	$-\frac{1103614848}{5}$	0	-3673875744
y^5	$-\frac{664048}{5}$	0	$-\frac{45638208}{5}$	0	$-\frac{1103614848}{5}$	0	$-\frac{28258348608}{25}$	0
y^6	0	$-\frac{47811456}{5}$	0	$-\frac{1490848128}{5}$	0	$-\frac{28258348608}{25}$	0	$-\frac{1133586703872}{25}$
y^7	$-\frac{127497216}{25}$	0	$-\frac{2008081152}{5}$	0	-3673875744	0	$-\frac{1133586703872}{25}$	0

6.2 Power series and measure associated

Recall that $\hat{E}$ is the formal group associated with E with respect to the parameter $t=-2x/y$ and $\lambda(t)$ denotes the formal logarithm of $\hat{E}$. We also fixed an embedding $i_p:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}_p$ such that the completion of K in $\mathbb{C}_p$ is $K_{\mathfrak{p}}$. Let W be the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$.

Definition 6.3. *We define $\widehat{\Theta}(s,t)$ to be the formal composition of the Laurent expansion of $\Theta(z,w)$ at the origin with $z=\lambda(s)$ and $w=\lambda(t)$*

$$\widehat{\Theta}(s,t)=\Theta(z,w)|_{z=\lambda(s),w=\lambda(t)}.$$

Let $z_0,w_0\in L\otimes\mathbb{Q}$ be torsion points whose order n is prime to p. Analogously we define $\widehat{\Theta}_{(z_0,w_0)}(s,t)$ and $\widehat{\Theta}^*_{(z_0,w_0)}(s,t)$

$$\widehat{\Theta}_{(z_0,w_0)}(s,t)=\Theta_{(z_0,w_0)}(z,w)|_{z=\lambda(s),w=\lambda(t)},$$
$$\widehat{\Theta}^*_{(z_0,w_0)}(s,t)=\widehat{\Theta}_{(z_0,w_0)}(s,t)-\langle w_0,z_0\rangle\delta_{z_0}s^{-1}-\delta_{w_0}t^{-1}.$$

The following theorem will allow us to construct p-adic measure. The assumption of p splitting prime is fundamental, in fact in the case of p supersingular the coefficients of $\widehat{\Theta}^*_{(z_0,w_0)}$ are p-adically unbounded.

Theorem 6.3. *Let $z_0, w_0 \in L \otimes \mathbb{Q}$ be torsion points whose order n is prime to p. Then we have that $\widehat{\Theta}^*_{(z_0,w_0)}$ is $\mathfrak{p}$-integral.*

$$\widehat{\Theta}^*_{(z_0,w_0)}(s,t) = \widehat{\Theta}_{(z_0,w_0)}(s,t) - \langle w_0, z_0 \rangle \delta_{z_0} s^{-1} - \delta_{w_0} t^{-1} \in W[[s,t]].$$

Proof. See [BK10].2.3 □

Recall that by Prop. 4.2.1 we have an isomorphism of formal groups defined over $\hat{\mathcal{R}}_\infty$

$$\eta : \hat{E} \to \hat{\mathbb{G}}_m$$
$$S \mapsto \eta(t) = \Omega_\mathfrak{p} t + \cdots$$

which is of the form $\eta(t) = \exp(\lambda(t)\Omega_\mathfrak{p}) - 1$. We let $\iota(T) = \Omega_\mathfrak{p}^{-1} T + \cdots$ to be the inverse power series of $\eta(t)$.

Definition 6.4. *We let $\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1, T_2)$ to be the formal power series defined by*

$$\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1, T_2) = \widehat{\Theta}^*_{(z_0,w_0)}(s,t)|_{s=\iota(T_1), t=\iota(T_2)}.$$

Using this power series we then define its measure associated

Definition 6.5. *Let $z_0, w_0 \in L \otimes \mathbb{Q} - L$ be torsion points of order prime to p. We define the measure μ_{z_0,w_0} on $\mathbb{Z}_p^2$ to be the measure associated to the power series $\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1, T_2)$ by Lemma 5.1.3.*

Lemma 6.2.1. *For any $z_0, w_0 \in (L \otimes \mathbb{Q}) - L$ be torsion points of order prime to p we have*

$$\int_{\mathbb{Z}_p^2} x^{k-1} y^j \, d\mu_{z_0,w_0}(x,y) = (-1)^{j+k-1} \Omega_\mathfrak{p}^{-j-k+1} (k-1)! \frac{E_{j,k}(z_0, w_0)}{A^j}$$

for integers $j \geq 0$ and $k > 0$.

Proof. First of all recall that by Lemma 5.1.5, if $\mu_{j,k}$ denotes the measure associated with the power series $D_1^j D_2^k \widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1, T_2)$ we have

$$\int_{\mathbb{Z}_p^2} d\mu_{j,k}(x,y) = \int_{\mathbb{Z}_p^2} x^k y^j \, d\mu_{z_0,w_0}.$$

By equation (5.2) we obtain

$$\int_{\mathbb{Z}_p^2} x^k y^j \, d\mu_{z_0,w_0} = \int_{\mathbb{Z}_p^2} \binom{j}{0} \binom{k}{0} d\mu_{j,k}(x,y) = D_1^j D_2^k \widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1, T_2)|_{(0,0)}.$$

In the proof of Lemma 5.2.2 we proved the following equality

$$(1 + T_1) \frac{d}{dT_1} f(T_1) = \left[(\Omega_\mathfrak{p} \lambda'(T))^{-1} \frac{d}{dT} f(T) \right] |_{T=\iota(T_1)},$$

that implies

$$D_1^j D_2^k \widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1, T_2)|_{(0,0)} = \Omega_\mathfrak{p}^{-j-k} \partial_z^k \partial_w^j \Omega_{z_0,w_0}(z,w)|_{(0,0)}.$$

By the Laurent expansion of Theorem 6.2 we then conclude

$$\int_{\mathbb{Z}_p^2} x^{k-1} y^j \, d\mu_{z_0,w_0}(x,y) = (-1)^{j+k-1} \Omega_\mathfrak{p}^{-j-k+1} (k-1)! \frac{E_{j,k}(z_0, w_0)}{A^j}$$

□

Observe that when $z_0, w_0 \in L$ then we cannot calculate directly the interpolation property of μ_{z_0,w_0}. due to the factors s^{-1} and t^{-1} subtracted in the definition of $\widehat{\Theta}^*_{(z_0,w_0)}(s,t)$ instead of z^{-1} and w^{-1}. To deal with this we consider the restriction to $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. Applying the restriction given by Lemma 5.1.7 to both variables we obtain that the measure associated with the power series

$$\begin{aligned}\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1,T_2) - \frac{1}{p}\sum_{\zeta^p-1}\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(\zeta(1+T_1)-1,T_2) - \\ -\frac{1}{p}\sum_{\zeta^p-1}\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(T_1,\zeta(1+T_2)-1) + \\ +\frac{1}{p^2}\sum_{\substack{\zeta_1^p-1\\ \zeta_2^p-1}}\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(\zeta_1(1+T_1)-1,\zeta_2(1+T_2)-1)\end{aligned}$$

is the restriction of the measure μ_{z_0,w_0}. Furthermore, we can observe that this power series coincide with the one given by $\widehat{\Theta}^\iota$ instead of $\widehat{\Theta}^{*\iota}$. Indeed, the poles coming from $\iota(T_1)^{-1}$ and $\iota(T_2)^{-1}$ cancels out, in particular the terms involving $\iota(T_1)^{-1}$ add up as

$$\frac{1}{\iota(T_1)} - \frac{1}{p}\sum_{\zeta^p-1}\frac{1}{\iota(\zeta(1+T_1)-1)} - \frac{1}{p}\sum_{\zeta^p-1}\frac{1}{\iota(T_1)} + \frac{1}{p^2}\sum_{\substack{\zeta_1^p-1\\ \zeta_2^p-1}}\frac{1}{\iota(\zeta_1(1+T_1)-1)} = 0.$$

We can rewrite the power series associated to the restricted measure as

$$\begin{aligned}\widehat{\Theta}^{\iota}_{(z_0,w_0)}(T_1,T_2) - \frac{1}{p}\sum_{\zeta^p-1}\widehat{\Theta}^{\iota}_{(z_0,w_0)}(\zeta(1+T_1)-1,T_2) - \\ -\frac{1}{p}\sum_{\zeta^p-1}\widehat{\Theta}^{\iota}_{(z_0,w_0)}(T_1\zeta(1+T_2)-1) + \\ +\frac{1}{p^2}\sum_{\substack{\zeta_1^p-1\\ \zeta_2^p-1}}\widehat{\Theta}^{\iota}_{(z_0,w_0)}(\zeta_1(1+T_1)-1,\zeta_2(1+T_2)-1).\end{aligned} \tag{6.8}$$

Prop. 6.2.1. *Let $\mathfrak{a}$ be an integral ideal of K prime to p and let v_0, w_0 be $\mathfrak{a}$-torsion points of $\mathbb{C}/L$. Let $s_{\mathfrak{p}^n}$ and $t_{\mathfrak{p}^n}$ be $\mathfrak{p}^n$-torsion points of the formal group $\hat{E}$ and let v_n, w_n be elements in $L \otimes \mathbb{Q}$ that respectively represents the images of $s_{\mathfrak{p}^n}$ and $t_{\mathfrak{p}^n}$. Let ϵ be an element of $\mathcal{O}_K$ such that*

$$\epsilon \equiv 1 \mod \mathfrak{p}^n, \quad \epsilon \equiv 0 \mod \overline{\mathfrak{p}}^n.$$

Then we have

$$\widehat{\Theta}_{\epsilon v_0,\epsilon w_0}(s[+]s_{\mathfrak{p}^n}, t[+]t_{\mathfrak{p}^n}) = \langle \epsilon v_n, \epsilon w_0\rangle_L \widehat{\Theta}_{\epsilon v_0+\epsilon v_n, \epsilon w_0+\epsilon w_n}(s,t).$$

Moreover, if $\epsilon \equiv 1 \mod \mathfrak{a}$ we have

$$\widehat{\Theta}_{\epsilon v_0,\epsilon w_0}(s[+]s_{\mathfrak{p}^n}, t[+]t_{\mathfrak{p}^n}) = \langle \epsilon v_n, \epsilon w_0\rangle_L \langle \epsilon w_n, (\epsilon-1)v_0\rangle_L \widehat{\Theta}_{v_0+\epsilon v_n, w_0+\epsilon w_n}(s,t)$$

Proof. See [BK10].2.20. □

Lemma 6.2.2. *Let $z_0, w_0 \in L \otimes \mathbb{Q}$ be such that $\mathfrak{a}z_0$ and $\mathfrak{a}w_0$ are in L and $N\mathfrak{a}$ is prime to p. Then*

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (1+T_1)^{x_1}(1+T_2)^{x_2} d\mu_{(\epsilon z_0, \epsilon w_0)}(x_1, x_2) =$$
$$= \Big[\Theta_{\epsilon z_0, \epsilon w_0}(z, w, L) - \Theta_{p\epsilon z_0, \epsilon w_0}(pz, w, \overline{\mathfrak{p}}L) - \Theta_{\epsilon z_0, p\epsilon w_0}(z, pw, \overline{\mathfrak{p}}L) +$$
$$+ \Theta_{p\epsilon z_0, p\epsilon w_0}(pz, pw, \overline{\mathfrak{p}^2}L)\Big] |_{\lambda(\iota(T_1)), \lambda(\iota(T_2))}$$

where $\epsilon \in \mathcal{O}_K$ is such that $\epsilon \equiv 1 \mod \mathfrak{p}$ and $\epsilon \equiv 0 \mod \overline{\mathfrak{p}}$. If in addition $\epsilon \equiv 0 \mod \mathfrak{a}$, then we have

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (1+T_1)^{x_1}(1+T_2)^{x_2} d\mu_{(z_0, w_0)}(x_1, x_2) =$$
$$= \Big[\Theta_{z_0, w_0}(z, w, L) - \Theta_{pz_0, \epsilon w_0}(pz, w, \overline{\mathfrak{p}}L) -$$
$$- \langle(\epsilon - 1)z_0, w_0\rangle_L \Theta_{\epsilon z_0, pw_0}(z, pw, \overline{\mathfrak{p}}L) +$$
$$+ \langle(\epsilon - 1)z_0, w_0\rangle_L \Theta_{p\epsilon z_0, p\epsilon w_0}(pz, pw, \overline{\mathfrak{p}}^2 L)\Big] |_{\lambda(\iota(T_1)), \lambda(\iota(T_2))}.$$

Proof. Recall that by definition we have

$$\widehat{\Theta}^\iota_{\epsilon z_0, \epsilon w_0}(T_1, T_2) = \Theta_{\epsilon z_0, \epsilon w_0}(z, w)|_{z=\lambda(\iota(T_1)), w=\lambda(\iota(T_2))}.$$

Then by Prop. 6.1.2 we have

$$\sum_{\zeta^p - 1} \widehat{\Theta}^\iota_{\epsilon z_0, \epsilon w_0}(\zeta(1+T_1) - 1, T_2) = \sum_{s_1 \in \widehat{E}[\pi]} \widehat{\Theta}_{\epsilon z_0, \epsilon w_0}(s[+]s_1, t) =$$
$$= \sum_{v_1 \in \mathfrak{p}^{-1}L/L} \langle \epsilon v_1, \epsilon w_0 \rangle_L \widehat{\Theta}_{\epsilon v_0 + \epsilon v_1, \epsilon w_0}(s, t)|_{s=\iota(T_1), t=\iota T_2} =$$
$$= \sum_{v_1 \in \mathfrak{p}^{-1}L/L} \langle \epsilon v_1, \epsilon w_0 \rangle_L \widehat{\Theta}_{\epsilon v_0 + \epsilon v_1, \epsilon w_0}(z, w)|_{z=\lambda(\iota(T_1)), w=\lambda(\iota(T_2))}$$

Applying the distribution relation Prop. 6.1.2 we obtain

$$\sum_{\zeta^p - 1} \widehat{\Theta}^\iota_{\epsilon z_0, \epsilon w_0}(\zeta(1+T_1) - 1, T_2) = p\Theta_{p\epsilon z_0, \epsilon w_0}(pz, w, \overline{\mathfrak{p}}L)|_{z=\lambda(\iota(T_1)), w=\lambda(\iota(T_2))}.$$

Similarly, we have

$$\sum_{\zeta^p - 1} \widehat{\Theta}^\iota_{\epsilon z_0, \epsilon w_0}(T_1, \zeta(1+T_2) - 1) = \Theta_{\epsilon z_0, p\epsilon w_0}(z, pw, \overline{\mathfrak{p}}L)|_{z=\lambda(\iota(T_1)), w=\lambda(\iota(T_2))},$$

$$\sum_{\substack{\zeta_1^p - 1 \\ \zeta_2^p - 1}} \widehat{\Theta}^\iota_{(z_0, w_0)}(\zeta_1(1+T_1) - 1, \zeta_2(1+T_2) - 1) = \Theta_{p\epsilon z_0, p\epsilon w_0}(pz, pw, \overline{\mathfrak{p}^2}L)|_{\lambda(\iota(T_1)), \lambda(\iota(T_2))}.$$

The first assertion now follows from the fact that the restricted measure is given by (6.8). For the last assertion see [BK10].3.5. ☐

6.3 Relation to Yager's p-adic measure

Let φ be a Hecke character of an imaginary quadratic field K with class number 1. Let $\mathfrak{f}$ be its conductor. Let Ω_∞ be a complex number such that $L = \Omega\mathfrak{f}$ is a period lattice of a Weierstrass integral model of E over $\mathcal{O}_K$. As usual, let p be a rational prime which splits as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ in K coprime with the discriminant $-d_K$. Fix a Weierstrass model of $\mathbb{C}/L$ over $\mathcal{O}_K$ and a p-adic period $\Omega_{\mathfrak{p}}$ of its formal group.

Definition 6.6. *We define the p-adic measure μ_ψ by*

$$\mu_\psi(x,y) = \sum_{\mathfrak{a}\in I_K(\mathfrak{f})/P_K(\mathfrak{f})} \mu_{\psi(\alpha_{\mathfrak{a}}\mathfrak{a})\Omega_\infty,0}(x, N\mathfrak{f}y, L),$$

where $\alpha_{\mathfrak{a}}$ is any element of $\mathfrak{a}^{-1}$ such that $\alpha_{\mathfrak{a}} \equiv 1 \mod {}^\times\mathfrak{f}$.

Observe that the definition does not depend on the choice of the representative $\mathfrak{a}$ and $\alpha_{\mathfrak{a}}$. This is because $\psi(\alpha) = \alpha$ if $\alpha \equiv 1 \mod {}^\times\mathfrak{f}$.

Recall that we define $L_\infty(\overline{\psi^{j+k}})$ to be

$$L_\infty(\overline{\psi}^{k+j}, k) = \left(1 - \frac{\psi(\mathfrak{p})^{k+j}}{N\mathfrak{p}^{j+1}}\right)\left(1 - \frac{\overline{\psi}(\overline{\mathfrak{p}})^{k+j}}{N\overline{\mathfrak{p}}^{k}}\right)\left(\frac{2\pi}{\sqrt{d_K}}\right)\Omega_\infty^{-(k+j)} L(\overline{\psi}^{k+j}, k).$$

The following theorem will now establish the connection between Yager's p-adic measure and the previously constructed one.

Theorem 6.4. *Let j, k be integers such that $k > -j \geq 0$. Then we have*

$$\frac{1}{\Omega_{\mathfrak{p}}^{j+k}} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} x^{k-1} y^j d\mu_\psi(x,y) = (-1)^{j+k-1}(k-1)! L_\infty(\overline{\varphi}^{k+j}, k)$$

Proof. Let $\varepsilon \in \mathcal{O}_K$ such that $\varepsilon \equiv 1 \mod \mathfrak{f}\mathfrak{p}$ and $\varepsilon \equiv 0 \mod \overline{\mathfrak{p}}$. Then by Lemma 6.2.2 we have

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} (1+T_1)^x (1+T_2)^y d\mu_{\psi(\alpha_{\mathfrak{a}}\mathfrak{a})\Omega_\infty,0}(x,y) =$$

$$= \Big[\Theta_{\psi(\alpha_{\mathfrak{a}}\mathfrak{a})\Omega_\infty,0}(z,w,L) - \Theta_{p\psi(\alpha_{\mathfrak{a}}\mathfrak{a})\Omega_\infty,0}(pz,w,\overline{\mathfrak{p}}L) - \Theta_{\epsilon\psi(\alpha_{\mathfrak{a}}\mathfrak{a})\Omega_\infty,0}(z,pw,\overline{\mathfrak{p}}L) +$$
$$+ \Theta_{p\epsilon\psi(\alpha_{\mathfrak{a}}\mathfrak{a})\Omega_\infty,0}(pz,pw,\overline{\mathfrak{p}^2}L)\Big]\,|_{\lambda(\iota(T_1)),\lambda(\iota(T_2))}.$$

By the homothety relation (6.4) we have that rescaling by a factor $\psi(\overline{\mathfrak{p}}) = p\psi(\mathfrak{p})^{-1}$ we obtain

$$\Theta_{p\psi(\alpha_{\mathfrak{a}}\mathfrak{a})\Omega_\infty,0}(pz,w,\overline{\mathfrak{p}}L) = p^{-1}\psi(\mathfrak{p})\Theta_{\psi(\alpha_{\mathfrak{a}\mathfrak{p}}\mathfrak{a})\Omega_\infty,0}(\psi(\mathfrak{p})z, p^{-1}\psi(\mathfrak{p})w, L).$$

By Theorem 6.2 we have

$$\partial_z^{k-1}\partial_w^j \Theta_{\psi(\alpha_{\mathfrak{a}\mathfrak{p}}\mathfrak{a})\Omega_\infty,0}(\psi(\mathfrak{p})z, p^{-1}\psi(\mathfrak{p})w, L)|_{z=0,w=0} = (-1)^{j+k-1}\frac{\psi(\mathfrak{p})^{k+j-1}}{p^j}\frac{(j-1)!}{A(L)^j}E_{j,k}(\psi(\alpha_{\mathfrak{a}}\mathfrak{a}\mathfrak{p})\Omega_\infty, 0, \overline{\mathfrak{p}}L)$$

Recall that by Corollary 3.5.1 we have

$$\sum_{\mathfrak{a}\in I_K(\mathfrak{f})/P_K(\mathfrak{f})} E_{j,k}(\psi(\alpha_{\mathfrak{a}}\mathfrak{a}\mathfrak{p})\Omega_\infty, 0, \overline{\mathfrak{p}}L) = |\Omega_\infty|^{2j}\Omega_\infty^{-(j+k)} N\mathfrak{f}^{-j} L(\overline{\psi}^{j+k}, k).$$

Since $L = \Omega_\infty\mathfrak{f}$, we have

$$A(L) = N\mathfrak{f}|\Omega_\infty|\frac{\sqrt{d_K}}{2\pi}$$

and so we conclude

$$\sum_{\mathfrak{a}\in I_K(\mathfrak{f})/P_K(\mathfrak{f})} \partial_z^{k-1}\partial_w^j \Theta_{\psi(\alpha_{\mathfrak{a}\mathfrak{p}}\mathfrak{a})\Omega_\infty,0}(pz,w,\overline{\mathfrak{p}})L)|_{z=0,w=0} =$$

$$= (-1)^{j+k-1}(j-1)!\left(\frac{2\pi}{\sqrt{d_K}}\right)^j \frac{\psi(\mathfrak{p})^{j+k}}{p^{j+1}}\frac{L_{\mathfrak{f}}(\overline{\psi}^{j+k}, k)}{\Omega_\infty^{j+k}}.$$

Similarly, we have

$$\sum_{\mathfrak{a}\in I_K(\mathfrak{f})/P_K(\mathfrak{f})} \partial_z^{k-1}\partial_w^j \Theta_{\epsilon\psi(\alpha_{\mathfrak{a}\mathfrak{p}}\mathfrak{a})\Omega_\infty,0}(z, pw, \overline{\mathfrak{p}}L)|_{z=0,w=0} =$$

$$= (-1)^{j+k-1}(j-1)!\left(\frac{2\pi}{\sqrt{d_K}}\right)^j \frac{\overline{\psi}(\overline{\mathfrak{p}})^{j+k}}{p^{j+1}} \frac{L_{\mathfrak{f}}(\overline{\psi}^{j+k}, k)}{\Omega_\infty^{j+k}}$$

and

$$\sum_{\mathfrak{a}\in I_K(\mathfrak{f})/P_K(\mathfrak{f})} \partial_z^{k-1}\partial_w^j \Theta_{p\epsilon\psi(\alpha_{\mathfrak{a}\mathfrak{p}}\mathfrak{a})\Omega_\infty,0}(pz, pw, \overline{\mathfrak{p}}^2L)|_{z=0,w=0} =$$

$$= (-1)^{j+k-1}(j-1)!\left(\frac{2\pi}{\sqrt{d_K}}\right)^j \frac{\psi(\mathfrak{p})^{j+k}\overline{\psi}(\overline{\mathfrak{p}})^{j+k}}{p^{j+k+1}} \frac{L_{\mathfrak{f}}(\overline{\psi}^{j+k}, k)}{\Omega_\infty^{j+k}}.$$

Since $\partial_{S,\log} = \Omega_{\mathfrak{p}}\partial_z$ and $\partial_{T,\log} = \Omega_{\mathfrak{p}}\partial_w$, the assertion now follows from the definition of μ_ψ and the equations above. □

Appendix A

Class Field Theory

For this section we mainly refer to [Sch], [Neu99] and [CF10].

Theorem A.1 (Local reciprocity law). *Let K a finite extension of $\mathbb{Q}_p$. For any L/K finite Galois extension, the subgroup $N_{L/K}(L^\times)$ is open in $K^\times$ and there exists a group isomorphism*

$$r_{L/K} : K^\times/N_{L/K}(L^\times) \xrightarrow{\sim} Gal(L/K)^{ab}$$

such that the following properties are satisfied

i) *if L/K is unramified, then $r_{L/K}(\pi_F) = Frob_{L/K}$,*

ii) *if L'/K' is a finite Galois extension with $K \subset K'$ and $L \subset L'$, the following diagram commutes*

$$\begin{array}{ccc} K'^\times/N_{L'/K'}(L'^\times) & \xrightarrow{N_{K'/K}} & K^\times/N_{L/K}(L^\times) \\ \downarrow{\scriptstyle r_{L'/K'}} & & \downarrow{\scriptstyle r_{L/K}} \\ Gal(L'/K')^{ab} & \longrightarrow & Gal(L/K)^{ab} \end{array}$$

where the bottom horizontal arrow is the morphism induced by the restriction map $Gal(L'/K') \to Gal(L/K)$

iii) *if $\tau : L \xrightarrow{\sim} L'$ is an automorphism of valued fields and if $K' = \tau(K)$, we have a commutative diagram*

$$\begin{array}{ccc} K^\times/N_{L/K}(L^\times) & \xrightarrow{\tau} & K'^\times/N_{L'/K'}(L'^\times) \\ \downarrow{\scriptstyle r_{L/K}} & & \downarrow{\scriptstyle r_{L'/K'}} \\ Gal(L/K)^{ab} & \longrightarrow & Gal(L'/K')^{ab} \end{array}$$

where the bottom horizontal arrow is the isomorphism of groups induced by $\sigma \mapsto \tau\sigma\tau^{-1}$.

Moreover, there is at most one family of isomorphisms $(r_{L/K})_{L/K}$ satisfying i) and ii).

Consider now L/K a finite abelian extension of number fields. The local reciprocity law allows us to define a group homomorphism for every place v of K

$$\begin{aligned} K_v^\times &\to Gal(L/K) \\ x_v &\mapsto (x_v, L/K) \end{aligned}$$

which is the composite of the following chain of homomorphisms

$$K_v \twoheadrightarrow K_v^\times/N(L_w/K_v)(L_w^\times) \xrightarrow{r_{L_w/K_v}} Gal(L_w/K_v) \hookrightarrow Gal(L/K).$$

We can therefore define a group homomorphism

$$\begin{aligned} Art_{L/K} : \mathbb{A}_K^\times &\to Gal(L/K) \\ (x_v)_v &\mapsto \prod_v (x_v, L/K). \end{aligned}$$

called Artin reciprocity map.

Theorem A.2 (Artin reciprocity map)**.** *The Artin reciprocity map $Art_{L/K}$ induces a group isomorphism*

$$Art_{L/K} : \mathbb{A}_K^\times / K^\times N_{L/K}(\mathbb{A}_L^\times) \xrightarrow{\sim} Gal(L/K).$$

Moreover $Art_{L/K}$ is the unique continuous group homomorphism from $\mathbb{A}_K^\times$ to Gal(L/K) such that, for all v unramified in L, we have

$$Art_{L/K}(\pi_v) = Frob_{L/K}$$

where $\pi_v = (1, \dots, 1, \pi_v, 1 \dots) \in \mathbb{A}_k^\times$ is the idele whose all coordinates are 1 excepted the coordinate at v which is the uniformizer.

Theorem A.3 (Takagi-Chevalley, Existence theorem)**.** *The map $L \mapsto K^\times K^\times N_{L/K}(\mathbb{A}_L^\times)$ induces a decreasing bijection between isomorphism classes of finite abelian extensions of K and open subgroup of finite index containing $K^\times$ in $\mathbb{A}_K^\times$.*

Theorem A.4. *Let $\mathfrak{m}$ be a modulus for a number field K. We have an exact sequence*

$$1 \to \mathcal{O}_K^\times/(\mathcal{O}_K^\times \cap K^{\mathfrak{m},1}) \to K^{\mathfrak{m}}/K^{\mathfrak{m},1} \to Cl_K^{\mathfrak{m}} \to Cl_K \to 1 \tag{A.1}$$

and a canonical isomorphism

$$K^{\mathfrak{m}}/K^{\mathfrak{m},1} \cong \{\pm\}^{\#\mathfrak{m}_\infty} \times (\mathcal{O}_K/\mathfrak{m}_0)^\times. \tag{A.2}$$

Corollary A.4.1. *Let $\mathfrak{m}$ be a modulus for K. The ray class group $Cl_K^{\mathfrak{m}}$ is a finite abelian group whose cardinality $h_K^{\mathfrak{m}}$ is given by*

$$h_k^{\mathfrak{m}} = h_K \frac{\varphi(\mathfrak{m})}{[\mathcal{O}_K^\times : \mathcal{O}_K^\times \cap K^{\mathfrak{m},1}]} \tag{A.3}$$

where $h_K = \#Cl_K$ and $\varphi(\mathfrak{m}) = \#(K^{\mathfrak{m}}/K^{\mathfrak{m},1}) = \varphi(\mathfrak{m}_\infty)\varphi(\mathfrak{m}_0)$, with

$$\varphi(\mathfrak{m}_\infty) = 2^{\#\mathfrak{m}_\infty}, \quad \varphi(\mathfrak{m}_0) = \#(\mathcal{O}_K/\mathfrak{m}_0)^\times = N(\mathfrak{m}_0) \prod_{\mathfrak{p}|\mathfrak{m}_0} (1 - N(\mathfrak{p})^{-1}).$$

APPENDIX B

Elliptic curves with Complex Multiplication

For this section, we will mainly refer to Rubin's article [Rub81]and Silverman's book [Sil94]. Fix a subfield F of $\mathbb{C}$ and an elliptic curve E defined over F.

Definition B.1. *We say E has complex multiplication over F if $End_F(E)$ is an order in an imaginary quadratic field, i.e., if $End_F(E) \neq \mathbb{Z}$.*

We assume from now on that E has complex multiplication by $\mathcal{O}_K$ ring of integers of a quadratic imaginary extension K. There is a unique isomorphism

$$[\cdot] : \mathcal{O}_K \xrightarrow{\sim} End_F(E)$$

such that for any invariant differential $\omega \in \Omega_E$ on E we have

$$[\alpha]^* \omega = \alpha \omega$$

for all $\alpha \in \mathcal{O}_K$. For simplicity, we identify $End_F(E)$ with $\mathcal{O}_K$. If $\mathfrak{a} \subseteq \mathfrak{O}_K$ is an integral ideal, we will write $E[\mathfrak{a}] = \cap_{\alpha \in \mathfrak{a}}$. As usual, consider the associated lattice $L \subseteq \mathbb{C}$ and the analytic morphism

$$\xi : \mathbb{C}/L \to E(\mathbb{C})$$

Then for every $\mathfrak{a}$ fractional ideal of K we have that $\mathfrak{a}^{-1}L$ is a lattice in $\mathbb{C}$ and the associated elliptic curve $\mathfrak{a} * E = E_{\mathfrak{a}^{-1}L}$ has complex multiplication by $\mathcal{O}_K$. Furthermore, with the standard identification we have that to every $\alpha \in \mathcal{O}_K$ corresponds the morphism $\xi(z) \mapsto \xi(\alpha z)$.

Prop. B.0.1. *Let E elliptic curve with complex multiplication by $\mathcal{O}_K$.*

(i) *$E[\mathfrak{a}]$ is the kernel of the natural map $E \to \mathfrak{a} * E$*

(ii) *$E[\mathfrak{a}]$ is a free $\mathcal{O}_K/\mathfrak{a}$-module of rank 1.*

Corollary B.0.1. *Let E elliptic curve with complex multiplication by $\mathcal{O}_K$.*

(i) *For all integral ideals* $\mathfrak{a} \subseteq \mathcal{O}_K$*, the natural map* $E \to \mathfrak{a} * E$ *has degree* $N_{K/\mathbb{Q}}\mathfrak{a}$ *and* $\#E[\mathfrak{a}] = N_{K/\mathbb{Q}}\mathfrak{a}$*.*

(ii) *For all* $\alpha \in \mathcal{O}_K$*, the isogeny* $[\alpha]$ *has degree* $|N_{K/\mathbb{Q}}\alpha|$*.*

Theorem B.1 (Main theorem of complex multiplication)**.** *Let* $\mathfrak{a}$ *a fractional ideal of* K *and an analytic isomorphism*

$$\xi : \mathbb{C}/\mathfrak{a} \to E(\mathbb{C}).$$

Suppose $\sigma \in Aut(\mathbb{C}/K)$ *and* $x \in \mathbb{A}_K^\times$ *satisfies* $(x, K^{ab}/K) = \sigma|_{K^{ab}}$*. Then there is a unique isomorphism* $\xi' : \mathbb{C}/x^{-1}\mathfrak{a} \to E^\sigma(\mathbb{C})$ *such that the following diagram commutes*

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\xi} & E_{tors} \\ {\scriptstyle x^{-1}}\downarrow & & \downarrow{\scriptstyle \sigma} \\ K/x^{-1}\mathfrak{a} & \xrightarrow{\xi'} & E^\sigma_{tors} \end{array} \tag{B.1}$$

where E_{tors} *denotes the torsion in* $E(\mathbb{C})$ *and similarly for* E^σ_{tors}*.*

Let H denote the Hilbert class field of K.

Corollary B.1.1. (i) $K(j(E)) = H \subset F$*,*

(ii) $j(E)$ *is an integer of* H*.*

Corollary B.1.2. *There is an elliptic curve defined over* H *with endomorphism ring* $\mathfrak{O}_{\mathfrak{K}}$

Definition B.2. *Consider a Weierstrass equation for* E *over* H

$$y^2 = x^3 + Ax + B$$

with $A, B \in H$*. Then we define the Weber function for* E/H *as*

$$h(P) = h(x, y) = \begin{cases} x & \text{if } AB \neq 0, \\ x^2 & \text{if } B = 0, \\ x^3 & \text{if } A = 0. \end{cases}$$

Theorem B.2. *Let* K *imaginary quadratic field, let* E *be an elliptic curve with complex multiplication by* $\mathcal{O}_K$*, and let* $h : E \to \mathbb{C}$ *be the Weber function for* E/H*. Let* $\mathfrak{c}$ *be an integral ideal of* $\mathcal{O}_K$*. Then the field*

$$K(j(E), h(E[\mathfrak{c}])) = H(h(E[\mathfrak{c}]))$$

is the ray class field of K *modulo* $\mathfrak{c}$*.*

Corollary B.2.1. $K^{ab} = K(j(E), h(E_{tors}))$*. If in particular,* K *has class number 1,* $K = H$*, then* $K^{ab} = K(E_{tors})$*.*

Theorem B.3. *There is a Hecke character*

$$\psi = \psi_E : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times \tag{B.2}$$

with the following properties.

(i) *If* $x \in \mathbb{A}_F^\times$ *and* $y = N_{F/K}x \in \mathbb{A}_K^\times$*, then*

$$\psi(x)\mathcal{O}_K = y_\infty^{-1}(y\mathcal{O}_K).$$

(ii) *If* $x \in \mathbb{A}_F^\times$ *is a finite adele* ($x_\infty = 1$) *and* $\mathfrak{p}$ *is a prime of* K*, then* $\psi(x)(N_{F/K}x)_\mathfrak{p}^{-1} \in \mathcal{O}_{K_\mathfrak{p}}^\times$ *and for every* $P \in E[\mathfrak{p}^\infty]$

$$(x, F^{ab}/F)P = \psi(x)(N_{F/K}x)_\mathfrak{p}^{-1}P.$$

(iii) *If* $\mathfrak{q}$ *is a prime of* F*, then* ψ *is unramified at* $\mathfrak{q}$ *if and only if* E *has good reduction at* $\mathfrak{q}$.

Let $\mathfrak{f} = \mathfrak{f}_E$ denote the conductor of the Hecke character ψ_E. We can view ψ as a character of fractional ideals of F prime to $\mathfrak{f}$ in the usual way.

Corollary B.3.1. *As a character on ideals,* ψ *satisfies the following properties.*

(i) *If* $\mathfrak{b}$ *is an ideal of* F *prime to* $\mathfrak{f}$ *then*

$$\psi(\mathfrak{b})\mathcal{O}_K = N_{F/K}\mathfrak{b}.$$

(ii) *If* $\mathfrak{q}$ *is a prime of* F *not dividing* $\mathfrak{f}$ *and* $\mathfrak{b}$ *is an ideal of* $\mathcal{O}_K$ *prime to* $\mathfrak{q}$*, then for every* $P \in E[\mathfrak{b}]$ *we have*

$$(\mathfrak{q}, F(E[\mathfrak{b}])/F)P = \psi(\mathfrak{q})P.$$

(iii) *If* $\mathfrak{q}$ *is a prime of* F *where* E *has good reduction and* $q = N_{F/\mathbb{Q}}\mathfrak{q}$ *then* $\psi(\mathfrak{q}) \in \mathcal{O}_K$ *reduces modulo* $\mathfrak{q}$ *to the Frobenius endomorphism* ψ_q *of* $\tilde{E}$.

Corollary B.3.2. *Suppose* E *is defined over* K*,* $\mathfrak{a}$ *is an ideal of* K *prime to* $6\mathfrak{f}$*, and* $\mathfrak{p}$ *is a prime of* K *not dividing* $6\mathfrak{f}$.

(i) $E[\mathfrak{a}\mathfrak{f}] \subset E(K(\mathfrak{a}\mathfrak{f}))$.

(ii) *The action of* $Gal(\mathbb{C}/K)$ *on* $E[\mathfrak{a}]$ *induces an isomorphism*

$$Gal(K(E[\mathfrak{a}])/K) \xrightarrow{\sim} (\mathcal{O}_K/\mathfrak{a})^\times.$$

(iii) *If* $\mathfrak{b}|\mathfrak{a}$ *then the natural map*

$$Gal(K(\mathfrak{a}\mathfrak{f})/K(\mathfrak{b}\mathfrak{f})) \to Gal(K(E[\mathfrak{a}])/K(E[\mathfrak{b}])).$$

(iv) $K(E[\mathfrak{a}\mathfrak{p}^n])/K(E[\mathfrak{a}])$ *is totally ramified above* $\mathfrak{p}$.

(v) *If the map* $\mathcal{O}_K^\times \to (\mathcal{O}/\mathfrak{a})^\times$ *is injective then* $K(E[\mathfrak{a}\mathfrak{p}^n])/K(E[\mathfrak{a}])$ *is unramified outside of* $\mathfrak{p}$.

APPENDIX C

GP/Pari Code

```
EllQI1(d)={

  if(d%4==1, tau=(1+sqrt(d))/2,tau = sqrt(d)); \\ tau generator of ring of integer Z+
    tau*Z

  j0 = round(ellj(tau));
  e = ellfromj(j0);
  E = ellinit(e);
    \\ elliptic curve over Q with End_Q(E)=O_K
  E = ellminimalmodel(E);

  L = ellperiods(E);
  w1 = ellperiods(E)[1];
  w2 = ellperiods(E)[2];

  tau = w1/w2;
}




Gk(k,tau)=
{
  V = mfcoefs(mfEk(k),100);
  Ekpol = truncate(Ser(V));
  q= exp(2*Pi*tau*I);
  Ek = subst(Ekpol,x, q);
  G = Ek*(2*Pi*I)^k/(k!)*bernfrac(k);
  return(G);
}


ThetafromL(L,z0,w0,n)=
{

  w1 = L[1];
  w2 = L[2];
```

```
  A = real((w1*conj(w2)-w2*conj(w1))/(2*Pi*I));

  sigmaL = ellsigma(L);   \\ power series of sigma up to precision n
  trunsigma = truncate(sigmaL);     \\ truncation of sigma

  sigmaxy = subst(trunsigma,x,x+y+z0+w0);     \\ polynomial approximation of PS of sigma(x+y)
  sersigmaxy = sigmaxy+O(x^n);    \\up to precision n in both coefficients
  polsigmaxy = SertoPol(sersigmaxy,n);

  sigmax = subst(trunsigma,x,x+z0);
  sigmay = subst(trunsigma,x,x+w0);
  invsigmax = truncate(sigmax^(-1)+O(x^n));     \\ polynomial app. inverse power series of sigma in x
  invsigmay = truncate(subst(sigmay^(-1),x,y)+O(y^n));  \\ polynomial app. inverse power series of sigma in y

  s2 = -Gk(2,tau)*w2^(-2); \\Weil chap 6 variation 2

  if(z0 == 0 && w0 == 0, polexpz0w0 = 1,              \\series expansion of exp((z*conj(w0)+w*conj(z0))/(-A))
    serexpz0w0 = exp((x*conj(w0)+y*conj(z0))/(-A))+O(x^n);    \\up to precision n in both coefficients
    polexpz0w0 = SertoPol(serexpz0w0,n);
  );

  if(z0 == 0 && w0 == 0, s2polexpz0w0 = truncate(exp(-s2*x*y)+O(x^n)),             \\ series expansion of exp((z*conj(w0)+w*conj(z0))/(-A))
    s2serexpz0w0 = exp(-s2*(x+z0)*(y+w0))+O(x^n);   \\up to precision n in both coefficients
    s2polexpz0w0 = SertoPol(s2serexpz0w0,n);
  );



  ThetaL = exp(-(z0*conj(w0)/A))*polexpz0w0*s2polexpz0w0*polsigmaxy*invsigmax*invsigmay;
  ThetaL = truncate(ThetaL*x*y + O(x^n));                        \\ vector of coefficients in x translated by 1
  V1 = Vec(ThetaL);
  W = Vec(0, #V1);
  for( i = 1, #V1, W[i] = O(y^n));
  V1 = truncate(V1 + W);
  M0 = matrix(#V1);

  for(i =1, #V1, M0[#V1-i+1,]=Vecrev(V1[i],#V1)); \\ matrix of coefficients of Theta of dimension 3*n-3

  M = M0[1..n,1..n];        \\ matrix of coefficients of Theta up to x^n*y^n
}
```

```
VecEk(L,n,tau)={

    w1 = L[1];
    w2 = L[2];

    VEk = Vec(0,n);

    for(i=1,n-1, if(i>1 && i%2==0, VEk[i+1]= Gk(i,tau)*w2^(-i)));

    return(VEk);

}

\\ Compute theta expansion using Jacobi triple product

Thetaexptrans(L,z0,n,m)={

    w1 = L[1];
    w2 = L[2];
    tau = w1/w2;

    A = real((w1*conj(w2)-w2*conj(w1))/(2*Pi*I));

    q = exp(2*Pi*I*tau);
    z = exp(2*Pi*I*(x+z0)/w2);

    thetaexp =0;
    Pq3 = 0;

    for(i = 0, m, Pq3 += (-1)^i*(2*i+1)*q^(i*(i+1)/2));

    for(j=-m, m, thetaexp += (-1)^j*z^j*q^(j*(j+1)/2));

    thetaexp = thetaexp*exp(Pi*I*(x+z0)/w2)/Pq3*w2/(2*Pi*I)+O(x^n);

    thetaexp = thetaexp*exp(conj(w2)/(2*A*w2)*(x+z0)^2);

    return(thetaexp);

}



SertoPol(f,n)={

  vecf = Vec(f+O(x^n));
  W = Vec(0,#vecf);
  for(i=1, #vecf, W[i] = O(y^n));
  polf = Polrev(truncate(vecf+W));

  return(polf);
}
```

```
ThetaJTP(L,z0,w0,n,m)={


  x;
  y;
  w1 = L[1];
  w2 = L[2];
  tau = w1/w2;

  Thetax = Thetaexptrans(L,z0,n,m);
  if(z0==0, Thetax = serchop(Thetax,1));
  Thetay = subst(Thetaexptrans(L,w0,n,m),x,y);
  if(w0==0, Thetay = serchop(Thetay,1));
  Thetaxy = Thetaexptrans(L,y+z0+w0,n,m);

  Theta2 = Thetaxy*Thetax^(-1)*Thetay^(-1);

  if(z0 == 0 && w0 == 0, polexpz0w0 = 1,              \\series expansion of exp((z*
    conj(w0)+w*conj(z0))/(-A))
    serexpz0w0 = exp((x*conj(w0)+y*conj(z0))/(-A))+O(x^n);    \\up to precision n in
    both coefficients
    polexpz0w0 = SertoPol(serexpz0w0,n);
  );


  Thetaz0w0 = exp(-z0*conj(w0)/A)*polexpz0w0*Theta2*x*y+O(x^n);

  V2 = Vec(truncate(Thetaz0w0));
  W = Vec(0, #V2);
  for( i = 1, #V2, W[i] = O(y^n));
  V2 = truncate(V2 + W);
  N0 = matrix(#V2);

  for(i =1, #V2, N0[#V2-i+1,]=Vecrev(V2[i],#V2)); \\ matrix of coefficients of Theta
    of dimension 3*n-3

  N = N0[1..n-1,1..n-1];        \\ matrix of coefficients of Theta up to x^n*y^n

        return(N);
}


NewtonSumsEk(L,d,n,r)={

  w1 = L[1];
  w2 = L[2];
  tau = w1/w2;

  Sr = matrix(d+2,r);
  M = matrix(d+2);
```

```

  for(i=0, n-1,
    for(j=0, n-1,

      if(gcd(i,j)%n,
      rho = i*w1+j*w2 ;
      M = ThetaJTP(L,rho/n,0,d+3,20);

      for(l=1,d+2,
        for(c=1,r,
          Sr[l,c]=Sr[l,c]+M[l,2]^c)))));

  return(Sr);

}
```

List of Symbols

K quadratic immaginary field

$\mathcal{O}_K$ ring of integers

$-d_K$ discriminant of K

p rational prime

$\mathfrak{p}, \mathfrak{q}$ prime ideals

$v_\mathfrak{p}$ normalized $\mathfrak{p}$-adic valuation

π generator of $\mathfrak{p}$

$K_\mathfrak{p}$ completion of K at $\mathfrak{p}$

$\mathcal{O}_\mathfrak{p}$ ring of integers of $K_\mathfrak{p}$

$\mathfrak{a}, \mathfrak{b}, \mathfrak{m}$ integral ideals

ω_K roots of unity of K

$\omega_\mathfrak{f}$ roots of unity $\equiv 1 \mod \mathfrak{f}$

$K(\mathfrak{m})$ Ray class field modulo $\mathfrak{m}$

h_K class number

$h_K^\mathfrak{m}$ Ray class number modulo $\mathfrak{m}$

E elliptic curve

$\mathfrak{f}$ conductor of E

ψ Hecke character of elliptic curve

L period lattice

$\wp$ Weierstrass $\wp$-function

E_α kernel of endomorphism α

$\widehat{E}$ formal group of elliptic curve

$\tilde{E}$ reduction of elliptic curve mod $\mathfrak{p}$

F_m field extension of K generated by $\overline{\pi}^{m+1}$-torsion points

$K_{n,m}$ field extension of F_m generated by π^n-torsion points

$\Phi_{m,\omega}$ completion of F_m at ω

$\Xi_{n,m,\omega}$ completion of $K_{n,m}$ at prime above ω

F_∞ union of F_m

K_∞ union of $K_{n,m}$

$U'_{n,m}$ units of $\Xi_{n,m,}$

$U_{n,m}$ units of $\Xi_{n,m,}$ congruent to 1

G_∞ Galois group of K_∞/K

c_α Coleman power series associated to α

$g_{m,b}$ logarithmic derivative of $c_{m,\beta}$

$\sigma(z, L)$ Weierstrass's σ-function

$\Delta(L)$ Ramanujan's Δ-function

$A(L)$ area of lattice L

$\eta(z, L)$ η-function

$\theta(z, L)$ fundamental θ-function

$\Theta_{E,\mathfrak{a}}$ rational function on E with respect to $\mathfrak{a}$

$\Theta(z, \mathfrak{a})$ complex L-ellpitic function with respect to $\mathfrak{a}$

$\Lambda_m(z, \mathfrak{a})$ product of Theta functions over set of representatives

$C'_{n,m}$ group of elliptic units

$e(\mu)$ elliptic unit associated to map μ

$\langle z, w\rangle_L$ pairing associated to lattice L

$H_k(z, w, s, L)$ Eisenstein-Kronecker-Lerch series

$H_k^*(z, w, s, L)$ restricted Eisenstein-Kronecker-Lerch series

$E_k^*(z, L)$ Eisenstein series

$E_{j,k}^*(z_0, w_0, L)$ Eisenstein-Kronecker number

$L(\psi, s)$ L-function associated to ψ

$L_{\mathfrak{m}}(\psi, s)$ partial L-function associated to ψ and $\mathfrak{m}$

$\binom{x}{k}$ binomial coefficient function

f_μ power series associated to measure μ

μ_f measure associated to power series f

$\Gamma_f^{(i_1,i_2)}$ Gamma transform

$\omega(x)$ Teichmuller character

$\langle x\rangle_L$ $1 + p\,\mathbb{Z}_p$ part of x

u topological generator of $1 + p\,\mathbb{Z}_p$

D_i pdifferential operator $(1 + T_i)\partial/\partial T_i$

η isomorphism between $\widehat{E}$ and $\widehat{\mathbb{G}_m}$

$\Omega_{\mathfrak{p}}$ $\mathfrak{p}$-adic period associated to η

$(,)_n$ Weil pairing of p^{n+1}-division points of L

$L_\infty(\psi^{k+j}, k)$ scaled L-value associated to ψ, k, j

$\mathcal{G}^{(i_1,i_2)}$ Yager's power series that interpolates L-values

$\Theta(z, w, L)$ Kronecker two variable theta function

$\Theta_{(z_0,w_0)}(z, w, L)$ translated Kronecker two variable theta function

$\widehat{\Theta}_{(z_0,w_0)}(s, t)$ formal composition of Laurent expansion of theta and logarithm

$\widehat{\Theta}^*_{(z_0,w_0)}(s, t)$ truncation of $\widehat{\Theta}(s, t)$

$\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(s, t)$ composition of $\widehat{\Theta}^*(s, t)$ and η^{-1}

$\mu_{(z_0,w_0)}$ measure associated to $\widehat{\Theta}^{*\iota}_{(z_0,w_0)}(s, t)$

μ_ψ Bannai Kobayashi measure corresponding to Yager's construction

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