p-adic Modular Forms & their Geometry

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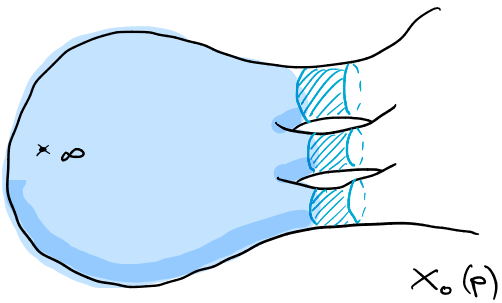

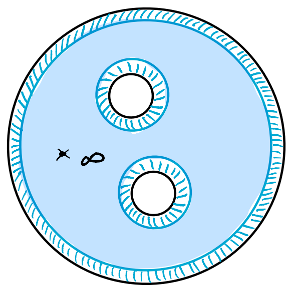

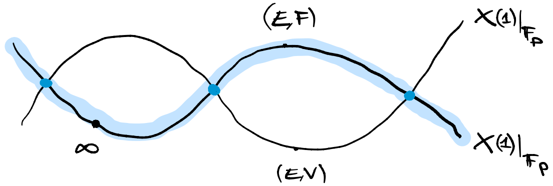

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§ I. Introduction:

Three Perspectives on Modular Forms

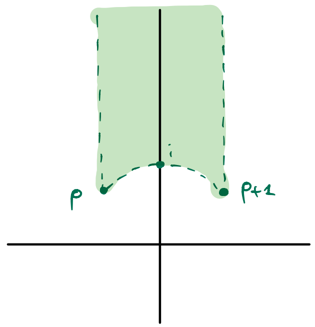
Complex Analytic

$f : \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic, $k \in \mathbb{Z}$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad z \in \mathcal{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \leq SL_2(\mathbb{Z})$$

(finite index)

+ extra conditions



Generating Series

$(q = e^{2\pi i z})$

– $G_k(q) = -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$, $\quad \sigma_{k-1}(n) = \sum_{1 \leq d | n} d^{k-1}$, $\quad B_k = -k \cdot \zeta(1-k)$

Eisenstein Series

Number theoretic flavour

– $\eta(q)^{-1} = q^{-\frac{1}{24}} \cdot \prod_{n \geq 1} (1-q^n)^{-1} = q^{-\frac{1}{24}} \cdot \sum_{n \geq 0} p(n) q^n$, $\quad p(n) = \#\{\text{partitions of } n\}$

Dedekind η-function

Combinatorial flavour

– $f \in S_2(\Gamma_0(N), \mathbb{Q})$ $\quad f(q) = \sum_{n \geq 1} a_n \cdot q^n$, $\quad a_p = p + 1 - \# E_f(\mathbb{F}_p)$ $\quad p \nmid N$

$E_f / \mathbb{Q}$ elliptic curve attached to f

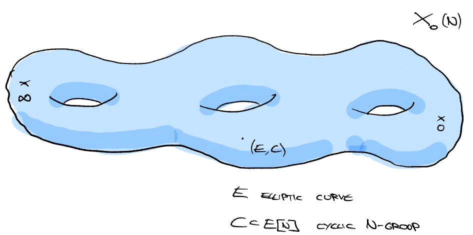
Weight 2 eigenform

Geometric flavour

Geometric

$f \in H^0(X_0(N), \underline{\omega}^k)$

Section of line bundle



Moduli space of elliptic curves w/ cyclic N-subgroup

CONGRUENCES BETWEEN MODULAR FORMS

- $\Delta(q) = q \cdot \prod_{n\geq 1} (1-q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n$ DISCRIMINANT MODULAR FORM
- $G_{12}(q) = \frac{691}{65520} + \sum_{n\geq 1} \sigma_{11}(n) q^n$ EISENSTEIN SERIES

RAMANUJAN'S CONGRUENCE ('16)

$$\tau(p) \equiv 1 + p^{11} \mod 691.$$

SERRE'S IDEA ('72):

CONSTRUCT P-ADIC ANALYTIC FAMILIES OF MODULAR FORMS FROM THEIR q-EXPANSION

$f(q) = a_0 + a_1 q + a_2 q^2 + \cdots \in \mathbb{C}_p[[q]]$, $\quad v_p(f) := \inf_n (v_p(a_n))$

THE SPACE OF **p-ADIC MODULAR FORMS** IS COLLECTION OF $f(q) \in \mathbb{C}_p[[q]]$ SUCH THAT

$$v_p(f(q) - f_i(q)) \longrightarrow \infty$$

FOR $f_i \in M_{k_i}(SL_2(\mathbb{Z}), \overline{\mathbb{Q}})$ CLASSICAL MODULAR FORMS

P-ADIC FAMILY OF EISENSTEIN SERIES ($p \geq 5$)

KUMMER CONGRUENCES ('51) LET $k \equiv k' \mod (p-1)p^n$

IF $(p-1) \nmid k$: $(1-p^{k-1})B_k/k \equiv (1-p^{k'-1})B_{k'}/k' \mod p^{n+1}$

CONSIDER $G_k^{(p)}(q) := (1-p^{k-1})\dfrac{-B_k}{2k} + \sum_{n\geq 1}\Big(\sum_{p\nmid d\mid n} d^{k-1}\Big)q^n$ MODULAR FORM FOR $\Gamma_0(p)$

$\rightsquigarrow$ BY FERMAT'S LITTLE THEOREM AND KUMMER CONGRUENCES FOR $k \equiv k' \mod (p-1)p^n$

- IF $(p-1)\nmid k$: $G_k^{(p)}(q) \equiv G_{k'}^{(p)}(q) \mod p^{n+1}$

LET $k \in \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$, $\sigma^*_{k-1}(n) = \sum_{p\nmid d\mid n} d^{k-1}$ P-ADIC INTEGER

$$G_k^{(p)} := \lim_{\overrightarrow{k_i}} G_{k_i}^{(p)} = \frac{1}{2}\zeta^*(1-k) + \sum_{n\geq 1}\sigma^*_{k-1}(n)q^n$$ IS A **P-ADIC MODULAR FORM**

$$\zeta^*(1-k) = \lim_{i\to\infty}\zeta(1-k_i)$$ $k_i \to k$, k_i INTEGERS

KUBOTA-LEOPOLD P-ADIC ζ-FUNCTION

Towards a finer description

Hecke Structure

(classical) Modular forms of level N are equipped w/ action of Hecke algebra

- $T_\ell f(q) = \sum_{n\geq 0} a_{\ell n} q^n + \ell^{k-1} \sum_{n \geq 0} a_n q^{\ell n}$, $\quad \ell \nmid N$
- $U_p f(q) = \sum_{n\geq 0} a_{np} q^n$, $\quad p \mid N$

U_p operator on p-adic modular forms has large continuous spectrum

$$\lambda \in p\mathbb{Z}_p, \qquad f_\lambda := (1-\lambda V_p)^{-1}(1 - V_p U_p) f$$

$$U_p f_\lambda = \lambda f_\lambda$$

Problem: We cannot decompose p-adic MF in finite linear combination eigenforms!

Katz's Idea ('72)

Introduce p-adic geometric framework:

Overconvergent Modular Forms

§ II. Geometry of Modular Curves

(p>5)

Complex Analytic vs Schemes

Moduli space of elliptic curves with cyclic p-subgroup

■ Complex picture

$\Gamma_0(p) = \left\{ \gamma \in SL_2\mathbb{Z} : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p \right\}$, $\quad Y_0(p) = \Gamma\backslash\mathcal{H} \longrightarrow \{\text{elliptic curves}/\mathbb{C} \text{ w/ p-subgp}\}$

$z \longmapsto (\mathbb{C}/(\mathbb{Z}+z\mathbb{Z}), \frac{1}{p})$

$X_0(p)$ its compactification is Riemann surface

■ Scheme theoretic picture

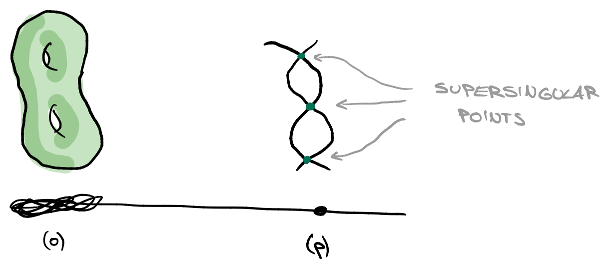
$\underline{\text{Rings}} \longrightarrow \underline{\text{Sets}}$

$R \longmapsto \{E/R \text{ elliptic curve w/ p-subgroup}\}/\simeq$

represented by smooth scheme $Y_0(p)$ over $\mathbb{Z}[1/p]$

take $X_0(p)$ its compactification, smooth proper over $\mathbb{Z}[1/p]$

Deligne – Rapaport model: semistable model over $\mathbb{Z}$



Connection with Modular Forms: $L/\mathbb{Q}$ field extension, $k \in \mathbb{Z}$

$S_{2k}(\Gamma_0(p), L) = H^0(X_0(p)_L, \Omega^{\otimes k}_{X_0(p)/L})$ cusp forms of weight 2k

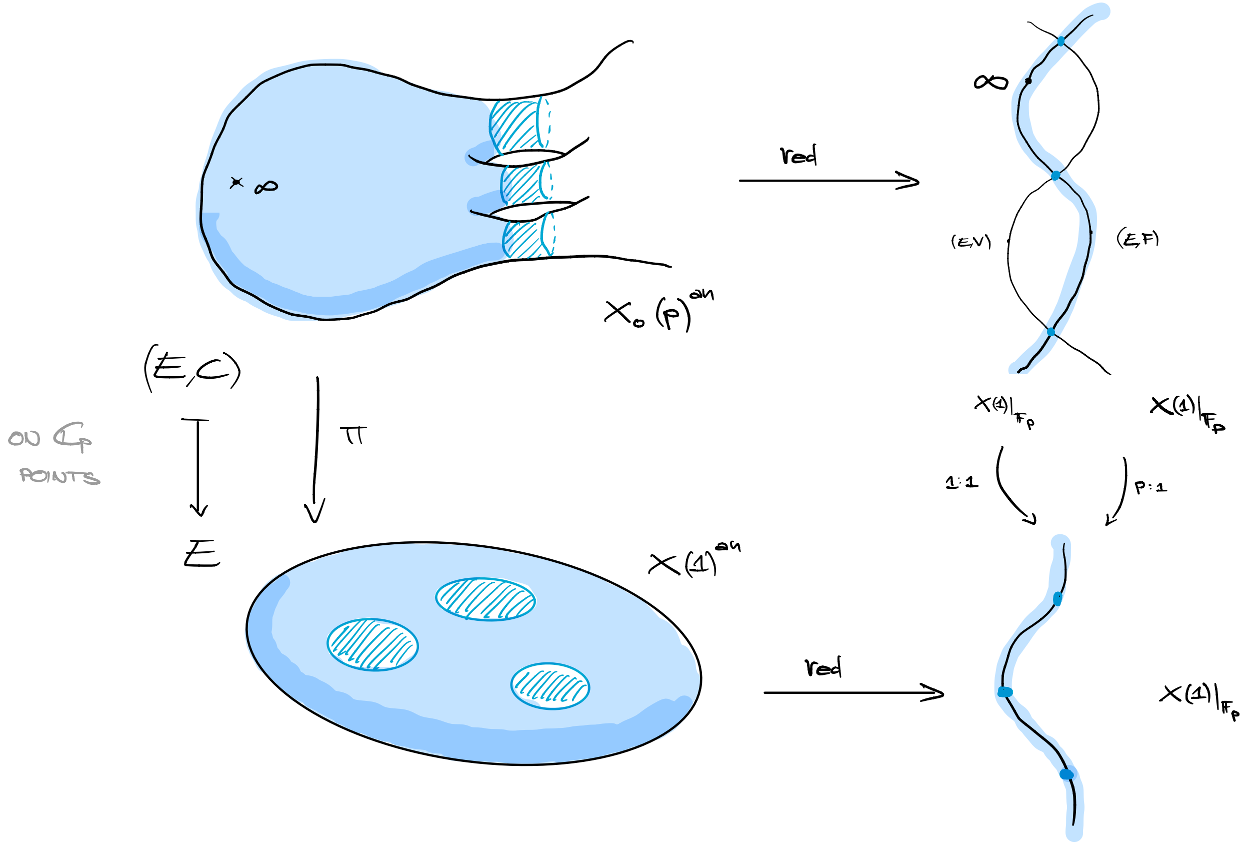
Rem. Taking modular curve over $\mathbb{C}_p$ will not add any new modular form $\Longrightarrow$ need Rigid Analytic Setting

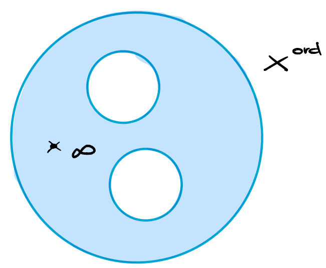
Rigid Picture: Ordinary Locus & Supersingular Annuli

Consider $E/\mathbb{F}_q$ elliptic curve over finite field

$E[p](\overline{\mathbb{F}}_q) \simeq \mathbb{Z}/p\mathbb{Z}$	Ordinary EC
$E[p](\overline{\mathbb{F}}_q) = \{0\}$	Supersingular EC

Rigid Analytification of modular curves





→ Ordinary Locus

→ Supersingular Locus

Achtung: A proper treatment should involve extra structure to avoid problems of representability of moduli problems.

The affinoid X^{ord} is called **Ordinary Locus** and corresponds to elliptic curves with ordinary reduction

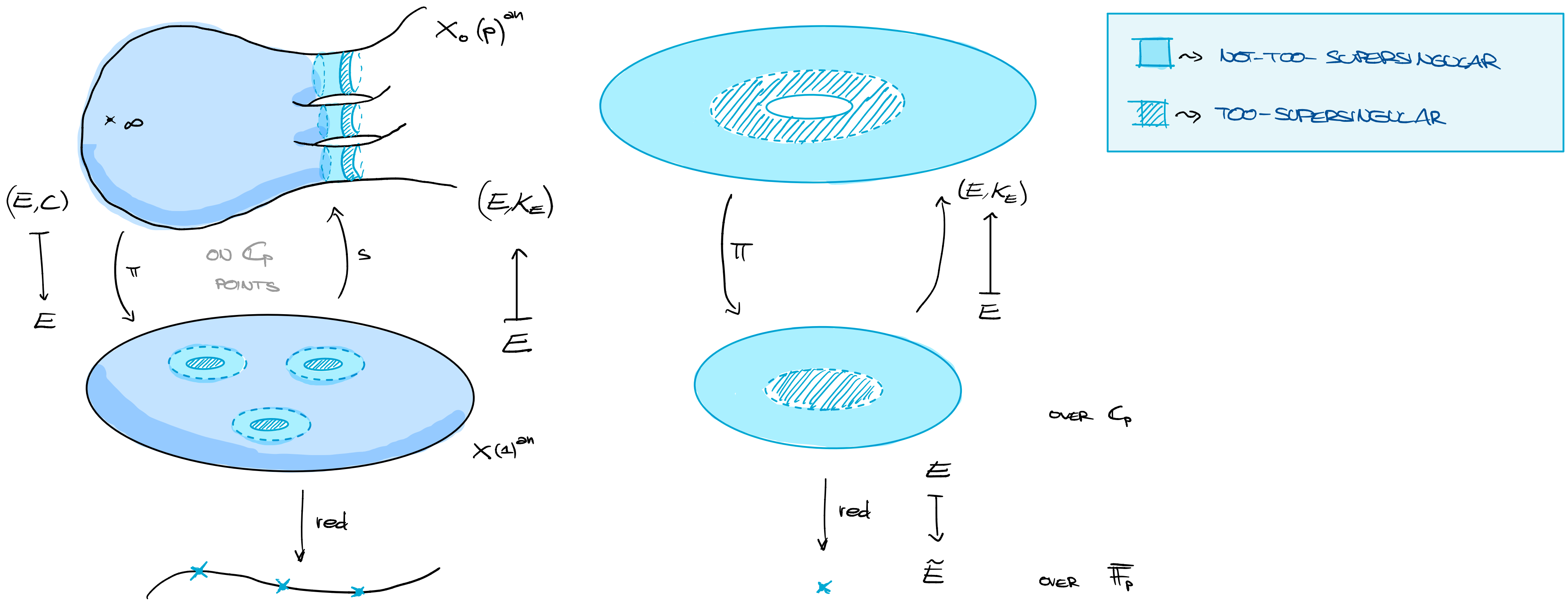
Rem p-adic modular forms à la Serre (of classical weights) correspond to sections over ordinary locus

$$S_{2k}^{(p)}(L) := \Omega^{\otimes k}_{rig}(X^{ord}_L), \qquad L/\mathbb{Q}_p \text{ field extension}$$

CANONICAL SUBGROUP: LUBIN'S IDEA

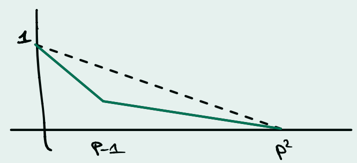
SUPERSINGULAR ANNULI HAVE SPECIAL LOCUS $\rightsquigarrow$ NOT TOO SUPERSINGULAR LOCUS

ASSOCIATE TO ELLIPTIC CURVE A CANONICAL SUBGROUP



WHERE DOES CANONICAL SUBGROUP COME FROM?

- ORDINARY LOCUS : $K_E = \mathrm{Ker}\left(E[p] \longrightarrow E[p](\overline{\mathbb{F}}_p)\right)$
- SUPERSINGULAR LOCUS : NEWTON POLYGON OF $[p]$ ON $\hat{E}$ FORMAL GROUP

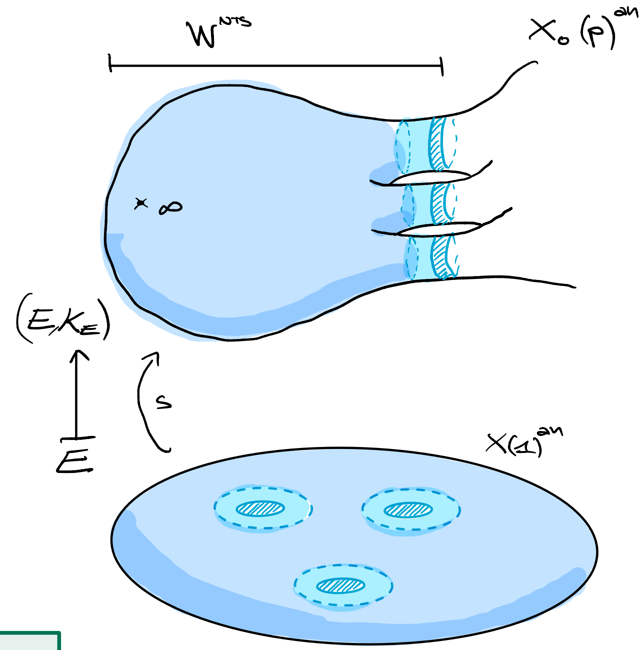


Overconvergent Modular Forms

Consider W^{NTS} the ordinary locus together with not-too-supersingular locus

$$X^{ord} \subset W^{NTS}$$

$W^{NTS} \setminus X^{ord}$ finite disjoint union of open annuli



An **Overconvergent p-adic Modular Form** is a p-adic modular form that is defined on W^{NTS} (overconverges in part of supersingular locus)

$$S^{\dagger}_{2k}(L) = \Omega^{\otimes k}_{rig}(W^{NTS}_{L}),$$ $L/\mathbb{Q}_p$ field extension.

Rem Classical modular forms of level $\Gamma_0(p)$ naturally sit in overconvergent p-adic modular forms

U_p OPERATOR : THE REASON EVERYTHING WORKS

RECALL U_p OPERATOR ON MODULAR FORMS $\quad U_p f(q) = \sum_{n \geq 1} a_{np} q^n, \quad f(q) = \sum_{n \geq 1} a_n q^n$

GIVEN BY GEOMETRIC CORRESPONDENCE

$$\begin{tikzcd}
 & X_0(Np) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X_0(N) & & X_0(N)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (E, \alpha, C) \arrow[dl] \arrow[dr] & \\
(E, \alpha) & & (E/C, \bar{\alpha})
\end{tikzcd}$$

~ PRESERVES ORDINARY LOCUS
NOT-TOO-SS LOCUS

THEOREM U_p IS A CONTINUOUS BOUNDED COMPACT OPERATOR ON S_{2k}^{+}

I.E. IMAGE OF UNIT BALL IS RELATIVELY COMPACT

SERRE AND DWORK ('62) PROVED FOLLOWING PROPERTIES OF THESE OPERATORS

- U_p IS LIMIT OF OPERATORS OF FINITE RANK
- U_p HAS WELL DEFINED TRACE
- U_p ADMITS A DISCRETE SPECTRUM

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq \cdots \quad w/ \quad |\lambda_i| \longrightarrow 0$$

AND SEQUENCE OF GENERALIZED EIGENFORMS f_i SUCH THAT EVERY OVERCONVERGENT MF f ADMITS

$$f \sim \sum \alpha_i f_i \quad \text{ASYMPTOTIC EXPANSION} \qquad \Big\| U^n f - \sum_{\geq \varepsilon} \alpha_i U^n f_i \Big\| = o(\varepsilon^n)$$

Made with Goodnotes

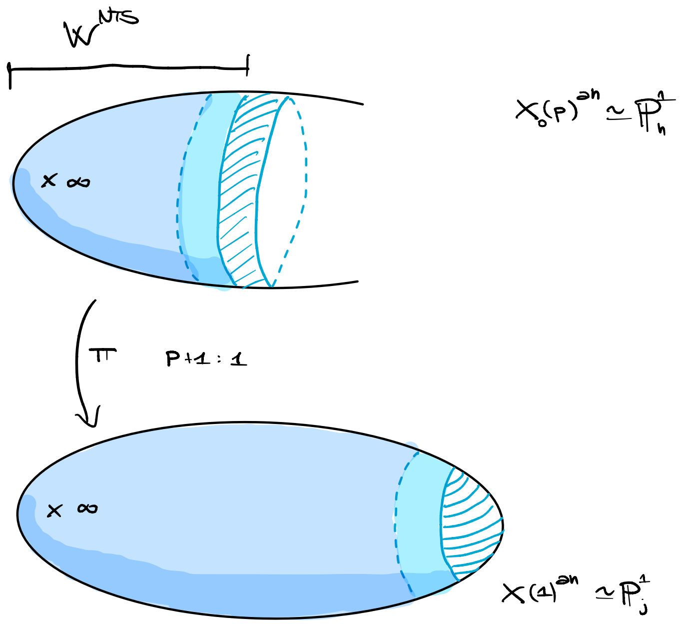
§ III. Computing U_p-Eigenfunctions

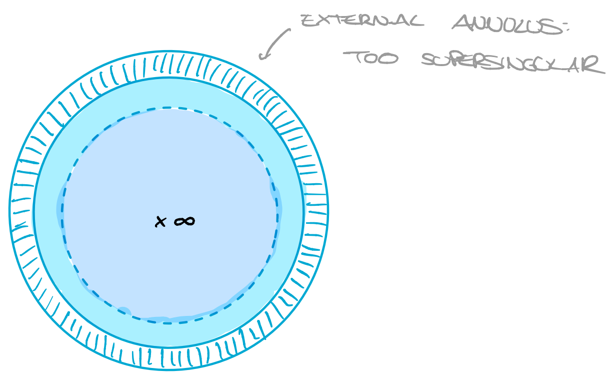
Genus 0 case

Let p be 2, 3, 5, 7 or 13.

$$h_p(z) = \left(\frac{\Delta(pz)}{\Delta(z)}\right)^{\frac{1}{p-1}}$$

Hauptmodul is uniformizer for $X_0(p)^{an}$





- $W^{NTS}(\mathbb{C}_p) \simeq \left\{ z \in \mathbb{C}_p \cup \{\infty\} \,\middle|\, |h_p(z)| < p^{\frac{12p}{p^2-1}} \right\} \simeq B\left(0, p^{12p/(p^2-1)}\right)$
- $X^{ord}(\mathbb{C}_p) \simeq \left\{ z \in \mathbb{C}_p \cup \{\infty\} \,\middle|\, |h_p(z)| \leq 1 \right\} \simeq B[0,1]$

Two rigid p-adic discs centered at the cusp ∞

> EG $p=2$, $j = \dfrac{(1+2^8 \cdot h_2)^3}{h}$

> The weight 0 overconvergent p-adic modular forms corresponds to function on p-adic disk W^{NTS}
> i.e Laurent series $f = a_0 + a_1 p^r h_p + a_2 p^{2r} h_p^2 + \dots \in \mathbb{C}_p[[p^r h_p]]$, $r = \dfrac{12p}{p^2-1}$ with growing conditions on coefficients

U_p ACTION ON HAUPTMODUL

FROM THE GEOMETRIC DESCRIPTION $\rightsquigarrow$ OVERCONVERGENT MODULAR FUNCTIONS EXPRESSED AS POWER SERIES ON h_p (+ GROWING CONDITIONS)

U_p ACTION ON $S_0^{\dagger}(\Gamma_p)$ GIVEN BY U_p ACTION ON $p^r h,\ p^{2r} h^2,\ p^{3r} h^3, \dots$ $\quad r = \frac{12\ p}{p^2-1}$

REM USE GENUS 0 OF $X_0(p)$ + DESCRIPTION OF U_p AS CORRESPONDENCE TO OBTAIN

$U_p h^i = P_i(h)$ $\quad P_i(x) \in \mathbb{C}_p[x]$ POLYNOMIAL $\rightsquigarrow$ COMPUTATIONALLY PRECISE

$P_i(h)$ CAN BE COMPUTED RECURSIVELY

EG $\quad p = 2$ $\qquad U_2(h^n) = (48h + 4096h^2)\, U_p(h^{n-1}) + h\, U_p(h^{n-2})$

$$v_2\big(U_2(i,j)\big)_{ij} = \begin{pmatrix}
3 & 8 & & & & & & & & \\
3 & 7 & 11 & 16 & & & & & & \\
 & 8 & 12 & 17 & 19 & 24 & & & & \\
 & 7 & 11 & 15 & 21 & 23 & 27 & 32 & & \\
 & & 11 & 19 & 20 & 25 & 27 & 35 & 35 & \cdots \\
 & & 11 & 16 & 20 & 24 & 27 & 33 & 35 & \\
 & & & 17 & 19 & 24 & 29 & 34 & 35 & \\
 & & & 15 & 20 & 23 & 27 & 31 & 38 & \\
 & & & & 19 & 24 & 27 & 37 & 36 & \\
 & & & & \vdots & & & & & \ddots
\end{pmatrix}$$

THEOREM (BUZZARD – CALEGARI, '05)

THE SLOPE SEQUENCE OF U_2 ON $S_2^{\dagger}(\Gamma_p)$ IS GIVEN BY

$$\left\{ 1 + 2 v_2\left(\frac{(3n)!}{n!} \right) \right\}_{n=1,\dots}$$

A Mysterious Behaviour

Consider $p=2$, let f_i eigenforms for U_2-operator, $U_2 f_i = \lambda_i f_i$ $\lambda_i \in \mathbb{C}_2$

$$v_2(\lambda_1) \leq v_2(\lambda_2) \leq v_2(\lambda_3) \leq \dots$$

Calegari ('13) observed that

q-expansion of f_{2^n} appear to converge to an infinite slope form

$$E_0^{[2]}(q) = \sum_{n\geq 1} \Big(\sum_{2 \nmid d \mid n} d^{-1} \Big) q^n$$

coming from integral of p-depletion of a classical modular form

50 years after Serre's first results, limits of q-expansions of p-adic modular forms reveal rich structures that remain largely mysterious!