HEIGHTS ON CURVES

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§ 1. Introduction

Fundamental tool required for study of rational and integral points on algebraic variety is a means of measuring size of point

K number field, V/K smooth projective variety
Fix embedding $V \subset \mathbb{P}^n$. A height function is a map

$$h : V(K) \longrightarrow [0, \infty).$$

Two important attributes:

- only finite number of points of bounded size ⇝ Mordell-Weil theorem
- size of a point should reflect arithmetic nature of point and geometry of variety ⇝ Gross-Zagier formula

Notation: K number field

M_K set of valuations
M_K^∞ archimedean valuations
M_K^0 non-archimedean valuations

$$v(x) = -\log \|x\|_v$$

Rem w/ suitable normalization of norms we have product formula

$$\prod_{v \in M_K} \|x\|_v = 1 \quad \forall x \in K^*.$$

§ 2. Naive heights

For rational point $P \in \mathbb{P}^n(\mathbb{Q})$ there is natural way to measure its size
write

$$P = (x_0 : x_1 : \cdots : x_n) \quad w/ \quad x_i \in \mathbb{Z}, \quad \gcd((x_i)_i) = 1$$

Define height of P to be the quantity

$$H(P) = \max\{|x_0|, \ldots, |x_n|\}$$

⇝ clear that for any B, $\{P \in \mathbb{P}^n(\mathbb{Q}) : H(P) \leq B\}$ is finite.

def K number field, $P = (x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n(K)$
the multiplicative height $H(P)$ of P is the quantity

$$H(P) = \left(\prod_{v \in M_K} \max\{\|x_0\|_v, \|x_1\|_v, \ldots, \|x_n\|_v\} \right)^{1/[K:\mathbb{Q}]}$$

the logarithmic height $h(P)$ of P is the quantity

$$h(P) = \log H(P) = \frac{1}{[K:\mathbb{Q}]} \left(\sum_{v \in M_K} -n_v \cdot \min\{v(x_0), v(x_1), \ldots, v(x_n)\} \right)$$

Rem by product formula height is independent of choice of homogeneous coordinates

THEOREM FOR ANY NUMBERS $B, D \geq 0$ THE SET

$$\{P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid H(P) \leq B \text{ AND } [\mathbb{Q}(P) : \mathbb{Q}] \leq D\} \text{ IS FINITE.}$$

THEOREM LET $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ RATIONAL MAP OF ~~deg~~ d DEFINED OVER $\overline{\mathbb{Q}}$
φ GIVEN BY $(n+1)$-TUPLE $\varphi = (f_0, \dots, f_m)$ HOM OF deg d.

LET Z SUBSET OF COMMON ZEROS OF f_i's, φ DEFINED ON $\mathbb{P}^n \setminus Z$

i) $h(\varphi(P)) \leq d \cdot h(P) + O(1) \quad \forall\, P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus Z$

ii) $X \hookrightarrow \mathbb{P}^n$ CLOSED SUBVARIETY OF $\mathbb{P}^n$, $X \cap Z = \emptyset$, THEN

$$h(\varphi(P)) = d \cdot h(P) + O(1) \quad \forall\, P \in X(\overline{\mathbb{Q}})$$

REM. $O(1)$ DEPEND ON φ BUT INDEPENDENT OF P.

def. LET $V/\overline{\mathbb{Q}}$ PROJECTIVE VARIETY, LET $\varphi : V \longrightarrow \mathbb{P}^n$ A MORPHISM.

THE LOGARITHMIC HEIGHT ON V RELATIVE TO φ IS THE FUNCTION

$$h_\varphi : V(\overline{\mathbb{Q}}) \longrightarrow [0, \infty), \qquad h_\varphi(P) := h(\varphi(P))$$

THEOREM (WEIL'S HEIGHT MACHINE)
LET k NUMBER FIELD, FOR EVERY SMOOTH PROJECTIVE VARIETY V/k THERE EXISTS A MAP

$$h_V : \mathrm{Div}(V) \longrightarrow \{\text{FUNCTIONS } V(\overline{k}) \longrightarrow \mathbb{R}\}$$

W/ FOLLOWING PROPERTIES

i) (NORMALIZATION)
LET $H \subset \mathbb{P}^n$ HYPERPLANE, $h(P)$ LOGARITHMIC HEIGHT ON $\mathbb{P}^n$. THEN

$$h_{\mathbb{P}^n, H}(P) = h(P) + O(1) \quad \text{FOR ALL } P \in \mathbb{P}^n(\overline{k})$$

ii) (FUNCTORIALITY)
LET $\varphi : V \longrightarrow W$ MORPHISM, $D \in \mathrm{Div}(W)$. THEN

$$h_{V, \varphi^* D}(P) = h_{W, D}(\varphi(P)) + O(1) \quad \text{FOR ALL } P \in V(\overline{k})$$

iii) (ADDITIVITY)
LET $D, E \in \mathrm{Div}(V)$ THEN

$$h_{V, D+E}(P) = h_{V,D}(P) + h_{V,E}(P) + O(1) \quad \text{FOR ALL } P \in V(\overline{k})$$

iv) (LINEAR EQUIVALENCE)
LET $D, E \in \mathrm{Div}(V)$ W/ D LINEARLY EQUIVALENT TO E. THEN

$$h_{V,D}(P) = h_{V,E}(P) + O(1) \quad \text{FOR ALL } P \in V(\overline{k}).$$

v) (POSITIVITY)

LET $D \in \mathrm{Div}(V)$ BE AN EFFECTIVE DIVISOR, B BASE LOCUS OF LINEAR SYSTEM $|D|$. THEN

$$h_{V,D}(P) \geq O(1) \quad \text{FOR ALL } P \in (V \setminus B)(\bar{k})$$

vi) (ALGEBRAIC EQUIVALENCE)

LET $D, E \in \mathrm{Div}(V)$ W/ D AMPLE, E ALGEBRAICALLY EQUIVALENT TO 0

THEN

$$\lim_{\substack{P \in V(\bar{k}) \\ h_{V,D}(P) \to \infty}} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0$$

vii) (FINITENESS)

LET $D \in \mathrm{Div}(V)$ AMPLE. THEN FOR EVERY FINITE EXTENSION k'/k AND EVERY CONSTANT B, THE SET

$$\{P \in V(k') \mid h_{V,D}(P) \leq B\} \quad \text{IS FINITE}$$

viii) (UNIQUENESS)

THE HEIGHT FUNCTIONS $h_{V,D}$ ARE DETERMINED UP TO $O(1)$ BY i), ii) AND iii).

REM $D \in \mathrm{Div}(V)$ W/ NO BASE POINTS $\varphi_D : V \longrightarrow \mathbb{P}^n$, $h_{V,D}(P) = h(\varphi_D(P))$

FOR GENERAL D, WRITE $D = D_1 - D_2$ W/ D_1, D_2 NO BASE POINTS

$$h_{V,D}(P) = h_{V,D_1}(P) - h_{V,D_2}(P).$$

COROL LET A/k BE AN ABELIAN VARIETY OVER NUMBER FIELD

$D \in \mathrm{Div}(A)$ BE A DIVISOR ON A.

i) LET m BE AN INTEGER, THEN FOR ALL $P \in A(\bar{k})$

$$h_{A,D}([m]P) = \frac{m^2+m}{2} h_{A,D}(P) + \frac{m^2-m}{2} h_{A,D}(-P) + O(1)$$

IN PARTICULAR, IF D SYMMETRIC $[-1]^* D \sim D$ THEN

$$h_{A,D}([m]P) = m^2 h_{A,D}(P) + O(1)$$

IF D ANTISYMMETRIC $[-1]^* D \sim -D$ THEN

$$h_{A,D}([m]P) = m\, h_{A,D}(P) + O(1)$$

ii) IF D HAS SYMMETRIC DIVISOR CLASS, THEN FOR ALL $P, Q \in A(\bar{k})$

$$h_{A,D}(P+Q) + h_{A,D}(P-Q) = 2h_{A,D}(P) + 2h_{A,D}(Q) + O(1) \quad \text{QUADRATIC FORM}$$

iii) IF D HAS ANTISYMMETRIC DIVISOR CLASS, THEN FOR ALL $P, Q \in A(\bar{k})$

$$h_{A,D}(P+Q) = h_{A,D}(P) + h_{A,D}(Q) + O(1) \quad \text{LINEAR FORM}$$

REM MUMFORD FORMULA $[m]^* D \sim \frac{m^2+m}{2} D + \frac{m^2-m}{2} [-1]^* D$

§ 3. Canonical Heights on Abelian Varieties

We can use previous corollary to kill the bounded function correction in the Weil Height machine.

Theorem (Néron, Tate)

Let A/k abelian variety defined over number field

$D \in \mathrm{Div}(A)$ divisor whose divisor class is symmetric $([-1]^* D \sim D)$

There is a height function

$$\hat{h}_{A,D} : A(\bar{k}) \longrightarrow \mathbb{R}$$

called canonical height on A relative to D w/ following properties

i) $\hat{h}_{A,D}(P) = h_{A,D}(P) + O(1) \qquad \forall P \in A(\bar{k})$.

ii) $\hat{h}_{A,D}([m]P) = m^2 \hat{h}_{A,D}(P) \qquad \forall P \in A(\bar{k}),\ \forall m$ integers

iii) (Parallelogram law)

$$\hat{h}_{A,D}(P+Q) + \hat{h}_{A,D}(P-Q) = 2\hat{h}_{A,D}(P) + 2\hat{h}_{A,D}(Q) \qquad \forall P, Q \in A(\bar{k})$$

iv) The canonical height map $\hat{h}_{A,D} : A(\bar{k}) \longrightarrow \mathbb{R}$ is a quadratic form

The associated pairing $\langle \cdot , \cdot \rangle_D : A(\bar{k}) \times A(\bar{k}) \longrightarrow \mathbb{R}$ defined by

$$\langle P, Q \rangle_D = \frac{\hat{h}_{A,D}(P+Q) - \hat{h}_{A,D}(P) - \hat{h}_{A,D}(Q)}{2}$$

is bilinear and satisfies $\langle P, P \rangle = \hat{h}_{A,D}(P)$.

v) (Uniqueness)

The canonical height $\hat{h}_{A,D}$ depends only on the divisor class of D

uniquely determined by (i), (ii) for any m

Rem Take

$$\hat{h}_{A,D}(P) = \lim_{n \to \infty} \frac{1}{4^n} h_{A,D}([2^n]P).$$

Corol Let $D \in \mathrm{Div}(A)$ be an ample divisor w/ symmetric divisor class.

i) For all $P \in A(\bar{k})$ we have

$$\hat{h}_{A,D}(P) \geq 0$$

w/ equality iff P point of finite order.

ii) The canonical height function extends $\mathbb{R}$-linearly to positive definite quadratic form

$$\hat{h}_{A,D} : A(\bar{k}) \otimes \mathbb{R} \longrightarrow \mathbb{R}$$

Rem There exists analogous construction for D antisymmetric

$\rightsquigarrow$ defines a linear form

THEOREM DEFINE A CANONICAL HEIGHT PAIRING BY THE FORMULA

$$[\cdot,\cdot]_A : A(\bar{k}) \times \mathrm{Pic}^\circ(A) \longrightarrow \mathbb{R}, \qquad [P,c]_A = \hat{h}_{A,c}(P)$$

i) THE CANONICAL HEIGHT PAIRING $[\cdot,\cdot]_A$ IS BILINEAR AND ITS KERNEL ON EITHER SIDE CONSISTS OF ELEMENTS OF FINITE ORDER

ii) LET $\hat{A}$ DUAL ABELIAN VARIETY, LET $\mathcal{P} \in A \times \hat{A}$ BE POINCARE DIVISOR CLASS THEN W/ NATURAL IDENTIFICATION OF $\hat{A}$ AND $\mathrm{Pic}^\circ(A)$

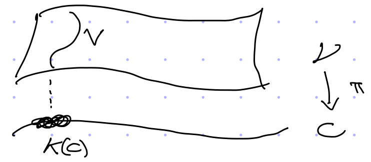
$$[P,c]_A = \hat{h}_{A\times\hat{A},\mathcal{P}}(P,c) \qquad \forall (P,c) \in A \times \hat{A}$$

§ 4 - INTERMEZZO (SPREADING HEIGHTS ON FUNCTION FIELDS)

V SMOOTH PROJECTIVE VARIETY DEFINED OVER $K = k(C)$ FUNCTION FIELD
C SMOOTH PROJECTIVE CURVE

CONSTRUCT PROJECTIVE VARIETY $\mathcal{V}/k$ W/ MORPHISM

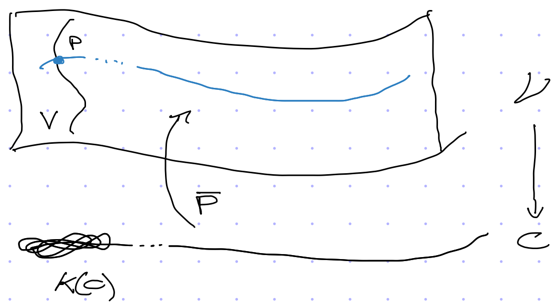
$$\pi : \mathcal{V} \longrightarrow C$$



SUCH THAT GENERIC FIBER OF π IS ISOMORPHIC TO V/K.

ASSUME $\mathcal{V}$ IS SMOOTH (HIRONAKA'S RESOLUTION OF SINGULARITIES)

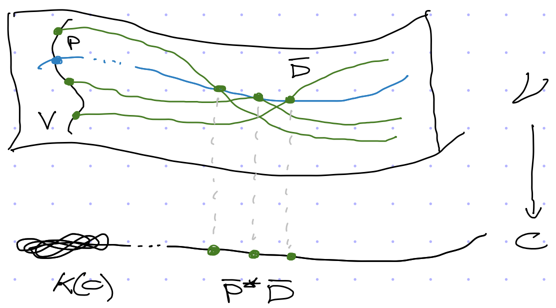
A POINT $P \in V(K)$ INDUCES A SECTION $\bar{P} : C \longrightarrow \mathcal{V}$



ANY DIVISOR $D = \sum n_Y \cdot Y$ ON V EXTENDS TO DIVISOR $\bar{D}$ ON $\mathcal{V}$ TAKING ZARISKI CLOSURE AND KEEPING SAME MULTIPLICITIES

$$\bar{D} := \sum n_Y \cdot \bar{Y}$$

$\rightsquigarrow$ $\bar{P}^*(\bar{D})$ WELL-DEFINED AS DIVISOR ON CURVE C IF $P \notin \mathrm{supp}\, D$



WE NOW DEFINE A FUNCTION $h_{D,\mathcal{V}}$ ON $V(K)$ BY THE FORMULA

$$h_{D,\mathcal{V}}(P) := \deg \overline{P}^* \overline{D} \qquad \text{FOR } P \in V(K)$$

WE WANT TO SHOW THAT $h_{D,\mathcal{V}}$ DEFINES A WEIL HEIGHT ASSOCIATED TO THE DIVISOR D.

LEMMA LET $Q = (f_0 : \cdots : f_n) \in \mathbb{P}^n(K)$, LET $\overline{Q}$ THE ASSOCIATED K-MORPHISM $\overline{Q} : C \longrightarrow \mathbb{P}^n$
H HYPERPLANE DIVISOR CLASS IN $\mathbb{P}^n$.

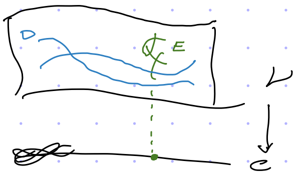
THEN

$$\deg \overline{Q}^* H = \sum_{P \in C} \max_{0 \le i \le n} (-\mathrm{ord}_P(f_i))$$

THIS IS HEIGHT $h_K(Q)$ OF K-RATIONAL POINT IN $\mathbb{P}^n$ FOR USUAL COLLECTION OF VALUATIONS ON ON FUNCTION FIELD K.

REM AN PRIME DIVISOR $Y \subset \mathcal{V}$ HAS IMAGE $\pi(Y)$ EITHER EQUAL TO C OR TO SINGLE POINT.
A DIVISOR D IS CALLED

- VERTICAL DIVISOR IF π MAPS ALL COMPONENTS TO POINTS
- HORIZONTAL DIVISOR IF π MAPS ALL COMPONENTS SURJECTIVLY TO C



D HORIZONTAL
E VERTICAL

LEMMA LET F VERTICAL DIVISOR ON $\mathcal{V}$. THEN THE MAP

$$h_{F,\mathcal{V}} : V(K) \longrightarrow \mathbb{Z}, \qquad P \longmapsto \deg \overline{P}^*(F)$$

TAKES FINITELY MANY VALUES. IN PARTICULAR, $h_{F,\mathcal{V}}$ IS BOUNDED FUNCTION

THEOREM FOR EVERY VARIETY V/K FIX A MODEL $\pi : \mathcal{V} \longrightarrow C$ AND FOR EVERY DIVISOR D ON V DEFINED OVER K, DEFINE A FUNCTION

$$h_{D,\mathcal{V}} = h_D : V(K) \longrightarrow \mathbb{Z}, \qquad h_D(P) = \deg \overline{P}^* \overline{D}$$

AS ABOVE. THEN

i) $h_{D+D'} = h_D + h_{D'}$

ii) LET $f \in K(V)^*$, $D = \mathrm{div} f$. THEN $h_D = O(1)$

iii) LET V/K, W/K VARIETIES, $\varphi : V \longrightarrow W$ BE A K-MORPHISM, D DIVISOR ON W DEFINED OVER K. THEN
$$h_D \circ \varphi = h_{\varphi^*(D)} + O(1)$$

iv) LET $H \subset \mathbb{P}^n$ HYPERPLANE DEFINED OVER K, h USUAL WEIL HEIGHT ON $\mathbb{P}^n(K)$.
THEN $h_H = h + O(1)$

v) IF D IS EFFECTIVE, $P \notin \mathrm{supp}(D)$, THEN $h_D(P) \geq 0$

REM THE ASSOCIATION $D \longmapsto h_D$ FROM DIVISORS TO FUNCTIONS SATISFIES AXIOMS OF A WEIL HEIGHT MACHINE

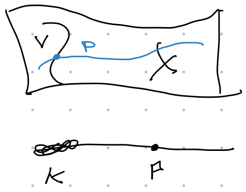
§5. Arakelov point of view

We want to build analogue of above construction when function field replaced by number field.

K number field
V/K smooth projective variety

$\pi : \mathcal{V} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ projective scheme w/ generic fiber V/K.

$P \in V(K)$ rational point gives a section $\bar{P} : \operatorname{Spec}(\mathcal{O}_K) \longrightarrow \mathcal{V}$



$$\begin{array}{c} \mathcal{V} \\ \downarrow \\ \operatorname{Spec} \mathcal{O}_K \end{array}$$

Problem: $\operatorname{Spec} \mathcal{O}_K$ is not complete!

$\rightsquigarrow$ need to take care of places at infinity

This leads to following definition

def A compactified (or Arakelov) divisor on $\operatorname{Spec} \mathcal{O}_K$ is a formal sum

$$E := \sum_{v \in M_K} m_v \cdot [v] \qquad \text{w/} \qquad m_v \in \begin{cases} \mathbb{Z} & \text{if } v \in M_K^0 \\ \mathbb{R} & \text{if } v \in M_K^\infty \end{cases}$$

A principal compactified divisor is a divisor of the form

$$\operatorname{div}(a) := \sum_{v \in M_K^0} \operatorname{ord}_v(a)[v] + \sum_{v \in M_K^\infty} -\log |a|_v \,[v] \qquad \text{for some } a \in K^*.$$

The degree of a compactified divisor $E = \sum m_v \cdot [v]$ is defined to be

$$\deg E := \sum_{v \in M_K^0} m_v \log N\mathfrak{p}_v - \sum_{v \in M_K^\infty} m_v [K_v : \mathbb{R}]$$

Rem Product formula implies degree of compactified divisor is zero.

The degree of an Arakelov divisor suggests the decomposition of the height into local pieces analogous to initial decomposition of the naive height.

We will state this decomposition in the case of abelian varieties

THEOREM (NÉRON)

LET A/k ABELIAN VARIETY DEFINED OVER A NUMBER FIELD. $D \in \mathrm{Div}\, A$ DIVISOR, THEN THERE IS A LOCAL HEIGHT FUNCTION

$$\hat{\lambda}_D : \coprod_{v \in M_k} A_D(k_v) \longrightarrow \mathbb{R}, \qquad A_D = A \setminus \mathrm{Supp}(D)$$

CALLED CANONICAL LOCAL HEIGHT ON A RELATIVE TO D, SATISFYING THE FOLLOWING CONDITIONS, W/ $\gamma_1, \gamma_2, \ldots$ M_k CONSTANTS

i) $\hat{\lambda}_{D_1+D_2,v} = \hat{\lambda}_{D_1,v} + \hat{\lambda}_{D_2,v} + \gamma_1(v)$

ii) IF $D = \mathrm{div}(f)$ THEN $\hat{\lambda}_{D,v} = v \circ f + \gamma_2(v)$

iii) IF $\varphi : B \longrightarrow A$ HOM. OF ABELIAN VARIETIES, THEN

$$\hat{\lambda}_{\varphi^* D,v} = \hat{\lambda}_{D,v} \circ \varphi + \gamma_3(v)$$

iv) LET $Q \in A(k)$ AND $\tau_Q : A \longrightarrow A$ BE TRANSLATION-BY-Q MAP. THEN

$$\hat{\lambda}_{\tau_Q^* D,v} = \hat{\lambda}_{D,v} \circ \tau_Q + \gamma_4(v)$$

v) LET $\hat{h}_D$ GLOBAL CANONICAL HEIGHT FUNCTION ON A RELATIVE TO D

$$\hat{h}_D(P) = \sum_{v \in M_k} \hat{\lambda}_{D,v}(P) \qquad \text{FOR ALL } P \in A_D(k)$$

§ 6. LOCAL HEIGHTS ON CURVES

LET $k = k_v$ LOCALLY COMPACT FIELD

X COMPLETE, SMOOTH, GEOMETRICALLY IRREDUCIBLE CURVE OF GENUS g

ASSUME IT HAS k-RATIONAL POINT

LET $\mathrm{Div}(X/k)$ SUBGROUP OF DIVISORS ON X RATIONAL OVER k
$\mathrm{Div}^0(X/k)$ SUBGROUP OF DEGREE 0
$Z^0(X/k)$ SUBGROUP OF $\mathrm{Div}^0(X/k)$ OF ELEMENTS OF DEG 0 IN FREE ABELIAN GROUP ON $X(k)$.
$P(X/k)$ SUBGROUP OF PRINCIPAL DIVISORS

LET J JACOBIAN OF X, ABELIAN VARIETY OF DIMENSION g DEFINED OVER k

FOR ANY EXTENSION FIELD H OF k WE HAVE $J(H) = \mathrm{Div}^0(X/H)/P(X/H)$.

PROP THERE IS UNIQUE FUNCTION $\langle a,b \rangle_v$ ON RELATIVELY PRIME DIVISORS $a \in Z^0(X/k_v)$, $b \in \mathrm{Div}^0(X/k_v)$ W/ VALUES IN $\mathbb{R}$ SATISFYING

i) $\langle a,b \rangle_v + \langle a,c \rangle_v = \langle a, b+c \rangle_v$

ii) $\langle a,b \rangle_v = \langle b,a \rangle_v$ WHENEVER $b \in Z^0(X/k_v)$

iii) $\langle a, \mathrm{div}(f) \rangle_v = \log |f(a)|_v$

iv) FIX b AND POINT $x_0 \in X(k_v) - \mathrm{Supp}(b)$ THEN $\big(X(k_v) - \mathrm{Supp}(b)\big) \longrightarrow \mathbb{R}$ DEFINED BY

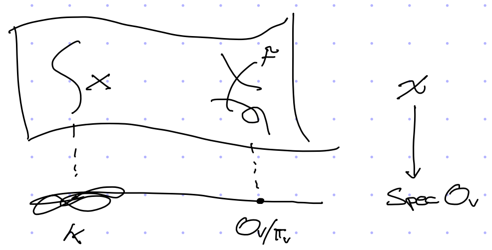
$$x \longmapsto \langle (x) - (x_0), b \rangle_v \quad \text{IS CONTINUOUS.}$$

- NON-ARCHIMEDEAN CASE

CONSIDER $\mathcal{X}/\mathcal{O}_v$ PROPER REGULAR MODEL FOR X OVER $\mathcal{O}_v$

$\mathcal{X}_k \simeq X$ GENERIC FIBER

LET $\mathcal{F} = \sum a_i \mathcal{F}_i$ DENOTE SPECIAL FIBRE, $\mathcal{F}_i$ IRREDUCIBLE COMPONENTS



def. LET D, E TWO EFFECTIVE DIVISORS ON $\mathcal{X}$ WITHOUT COMMON COMPONENT, $x \in \mathcal{X}$ CLOSED POINT CONTAINED IN BOTH

DEFINE THE INTERSECTION MULTIPLICITY AT x

$$i_x(D,E) := \mathrm{length}_{R_x}(R_x/(f,g))$$

W/ f,g EQUATIONS LOCALLY CUTTING OUT D,E IN NEIGHBORHOOD OF x.

DEFINE THEIR INTERSECTION

$$D \cdot E = \sum_x i_x(D,E)[x]$$

DEFINE THE TOTAL DEGREE OF INTERSECTION

$$(D \cdot E) = \deg D \cdot E = \sum_x i_x(D,E)$$

THEOREM LET $\mathrm{Div}_f(\mathcal{X})$ DENOTE GROUP OF FIBRAL DIVISORS.
THERE IS A UNIQUE BILINEAR PAIRING

$$\mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_f(\mathcal{X}) \longrightarrow \mathbb{Z}$$

EXTENDING ABOVE PAIRING

$$(D,E) \longmapsto (D \cdot E)$$

WHICH RESPECTS LINEAR EQUIVALENCE IN FIRST SLOT.

SINCE $\mathcal{X}_m = \mathcal{F} = \sum n_i \mathcal{F}_i$ IS PRINCIPAL DIVISOR $\mathrm{div}(\pi)$, WE GET

$(\mathcal{F}, \mathcal{F}_i) = 0 \quad \forall i$ $\rightsquigarrow$ WE CAN THEN DEDUCE SELF INTERSECTIONS $(\mathcal{F}_i^2)$.

TO DEFINE $\langle\cdot,\cdot\rangle_v$ LOCAL HEIGHT PAIRING, CONSIDER $D \in \mathrm{Div}^0(X/K_v)$, $E \in Z^0(X/K_v)$

LET $\mathcal{D}, \mathcal{E}$ ZARISKI CLOSURES OF D, E IN $\mathcal{X}$

THERE IS A $\mathbb{Q}$-DIVISOR $\mathcal{D}'$ ST $\mathcal{D}' - \mathcal{D}$ IS FIBRAL AND $\mathcal{D}' \cdot \mathcal{F}_i = 0 \ \forall i$
LIKEWISE FOR $\mathcal{E}$.

WE DEFINE $\langle D, E\rangle_v := -(\mathcal{D}' \cdot \mathcal{E}') \log q_v, \qquad q_v = \#(\mathcal{O}_v/\pi_v)$.

WHEN $D = (f)$ PRINCIPAL $(\mathcal{D} \cdot \mathcal{E}) = \sum n_i \, v(f(P_i)) = -\log_{q_v} |f(E)|_v$

- ARCHIMEDEAN CASE

ASSUME $K_v = \mathbb{C}$

THEN $M = X(K_v)$ IS RIEMANN SURFACE

A MEROMORPHIC DIFFERENTIAL $\omega \in \Gamma(M, \Omega^1_X \otimes K(X))$ IS SAID TO BE OF THE THIRD KIND IF

$$\operatorname{ord}_x(\omega) \geq -1 \quad \forall x \in M$$

THE DIVISOR

$$\operatorname{Res}(\omega) = \sum_x \operatorname{res}_x(\omega) \cdot (x)$$

HAS DEGREE ZERO BY GLOBAL RESIDUE THEOREM

REM. EVERY DIVISOR $D \in \operatorname{Div}^0(X/K_v)$ HAS THE FORM

$D = \operatorname{Res}(\omega)$ FOR ω DIFFERENTIAL OF THIRD KIND

$$0 \to \Omega^1_X \longrightarrow \Omega^1_X(\log|D|) \xrightarrow{\operatorname{Res}} \bigoplus_{P \in D} K_v \longrightarrow 0. \qquad H^1(M, \Omega^1_M) = H^0(M, \mathcal{O}_M)$$

ω_D IS UNIQUE UP TO ADDITION OF HOLOMORPHIC DIFFERENTIALS

$\leadsto$ WE MAY NORMALIZE IT INSISTING THAT ITS PERIODS BE PURELY IMAGINARY

IT THEN FOLLOWS

$$\omega_D + \overline{\omega}_D = d g_D \quad \text{IS EXACT.}$$

HERE g_D IS HARMONIC FUNCTION ON $M - \operatorname{Supp}(D)$ WELL DETERMINED UP TO ADDITION OF CONSTANT FUNCTION.

THIS IS THE GREEN'S FUNCTION ASSOCIATED TO DIVISOR D.

IF $D = \operatorname{div}(f)$ IS PRINCIPAL, THEN $\omega_D = df/f = d\log f$

$$g_D = \log|f|_v$$

IN GENERAL, IF m_x IS THE ORDER OF x IN D, THEN

$$g_D - m_x \log|z|_v \quad \text{IS HARMONIC NEAR } x$$

W/ z UNIFORMIZING PARAMETER

NOW ASSUME $E = \sum m_y (y)$ DIVISOR OF DEGREE ZERO ON $X(K)$

RELATIVELY PRIME TO D, DEFINE

$$\langle D, E \rangle_v = \sum m_y \cdot g_D(y).$$