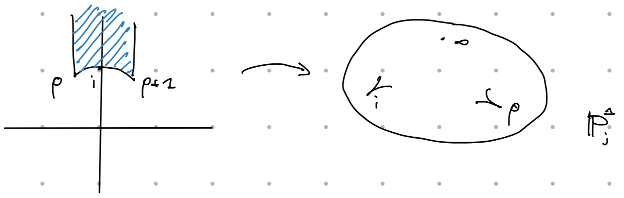
Modular Curves

- Complex picture
  - quotient of upper half plane
  - universal EC above
- Moduli theoretic picture
  - introduction to moduli problems and representability
  - $X(1)$ not rep. example
  - $\mathcal{E}ll$ category and moduli problems
  - generalized elliptic curves and compactifications

    (assume k - algeb. closed
  - $X_0(p)$ over $\mathbb{F}_p$
- q- expansions
  - classical description $/\mathbb{C}$
  - Tate curve
- Hecke operators
  - correspondences

§ 1. Classical complex picture

$Y = SL_2\mathbb{Z} \backslash \mathcal{H}$ $\qquad$ $PSL_2\mathbb{Z}$ gen. by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$



$S^2 = T^3 = id$ $\quad$ torsion elements

i, ρ $\quad$ elliptic points

more general $\rightsquigarrow$ $\Gamma(N) \leq \Gamma \leq SL_2\mathbb{Z}$ congruence subgroup

form $\quad Y(\Gamma) = \Gamma \backslash \mathcal{H}$

compactification given by adding action of Γ on $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$

We want to interpret these spaces as moduli spaces

especially interested in curves associated to

$$\Gamma_0(N) = \left\{ \gamma \in SL_2\mathbb{Z} : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \gamma \in SL_2\mathbb{Z} : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

- $Y_0(N) = \Gamma_0(N) \backslash \mathcal{H}$

  points are in bijection w/ iso classes of pairs (E, C)
  - E elliptic curve
  - C cyclic subgp order N

  for $z \in Y_0(N)$ $\rightsquigarrow$ $(E_z, C_z) = \left(\mathbb{C}/\mathbb{Z} + z\mathbb{Z}, \frac{1}{N}\mathbb{Z}/\mathbb{Z} \right)$

  Rem $N = 1$ $\quad Y(1)$ in bijection w/ class of EC $/\mathbb{C}$

  $\rightsquigarrow$ j invariant establishes bijection

  $Y(1) \longrightarrow \mathbb{C}$

- $Y_1(N) = \Gamma_1(N) \backslash \mathcal{H}$

  points are in bijection w/ iso classes of pairs (E, P)
  - E elliptic curve
  - P point of order N

FOR $z \in Y_0(N) \rightsquigarrow (E_z, P_z) = \left(\mathbb{C}/\mathbb{Z}+z\mathbb{Z}, \frac{1}{N} \bmod \mathbb{Z}+z\mathbb{Z}\right)$

ACTION OF $\Gamma_0(N)$ ON $\mathcal{H}$ INDUCES ACTION OF $\Gamma_0(N)/\Gamma_1(N)$ ON $Y_1(N)$

$$\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d$$

HAS MODULI - THEORETIC INTERPRETATION

$$\langle d \rangle : (E,P) \longmapsto (E, dP)$$ DIAMOND OPERATOR

NATURAL PROJECTION

$$\begin{aligned} Y_1(N) &\longrightarrow Y_0(N) \\ (E,P) &\longmapsto (E, \langle P \rangle) \end{aligned}$$

§2. MODULI THEORETIC PICTURE

CONSIDER $\underline{\mathrm{Sch}}$ THE CATEGORY OF SCHEMES

A CONTRAVARIANT FUNCTOR $F : \underline{\mathrm{Sch}} \longrightarrow \underline{\mathrm{Sets}}$ IS REPRESENTABLE IF THERE EXISTS SCHEME X SUCH THAT F IS ISOMORPHIC TO FUNCTOR

$$\begin{aligned} h^X : \underline{\mathrm{Sch}} &\longrightarrow \underline{\mathrm{Sets}} \\ T &\longmapsto \mathrm{Hom}_{\mathrm{Sch}}(T, X) \end{aligned}$$

EG

- STRUCTURE SHEAF

$$\begin{aligned} F : \underline{\mathrm{Sch}} &\longrightarrow \underline{\mathrm{Sets}} \\ X &\longmapsto \Gamma(X, \mathcal{O}_X) \end{aligned}$$

IS REPRESENTED BY $\mathbb{A}^1_{\mathbb{Z}}$

- MULTIPLICATIVE GROUP

$$\begin{aligned} F : \underline{\mathrm{Sch}} &\longrightarrow \underline{\mathrm{Sets}} \\ X &\longmapsto \Gamma(X, \mathcal{O}_X)^\times \end{aligned}$$

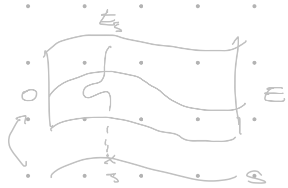
IS REPRESENTED BY $\mathbb{G}_m = \mathrm{Spec}\, \mathbb{Z}[T, T^{-1}]$

- ROOTS OF UNITY

$$\begin{aligned} F : \underline{\mathrm{Sch}} &\longrightarrow \underline{\mathrm{Sets}} \\ X &\longmapsto \{ a \in \Gamma(X, \mathcal{O}_X) \mid a^n = 1 \} \end{aligned}$$

IS REPRESENTED BY $\mu_n = \mathrm{Spec}\, \mathbb{Z}[T]/(T^n - 1)$.

A NAIVE DEFINITION OF MODULI SPACE OF CURVES



CONSIDER THE FUNCTOR $[\mathcal{E}\ell\ell]$

$$[\mathcal{E}\ell\ell] : \underline{\mathrm{Sch}} \longrightarrow \underline{\mathrm{Set}}$$
$$S \longmapsto \{ E/S \text{ ELLIPTIC CURVE} \}/\simeq_S$$

THIS FUNCTOR IS NOT REPRESENTABLE

- PRESENCE OF AUTOMORPHISMS
- BADLY BEHAVED UNDER FIELD EXTENSIONS $\sim$ TWISTS

SUPPOSE X REPRESENTS $[\mathcal{E}\ell\ell]$, THEN WE HAVE FUNCTORIAL ISOMORPHISM

$$\mathrm{Hom}_{\mathrm{Sch}}(S, X) \simeq [\mathcal{E}\ell\ell](S)$$

LET L/K FIELD EXTENSION

$\rightsquigarrow$ MAP $[\mathcal{E}\ell\ell](K) \longrightarrow [\mathcal{E}\ell\ell](L)$ SHOULD BE INJECTIVE

IN PARTICULAR IF TWO EC DEFINED OVER K BECOME ISOMORPHIC OVER L THEN THEY SHOULD HAVE BEEN ISOMORPHIC OVER K

(COUNTER) EG $\quad E_i : \; y^2 z = x^3 + (-1)^i z^3 \qquad i \in \{1,2\}$

ISOMORPHIC OVER $\overline{\mathbb{Q}}$ BY

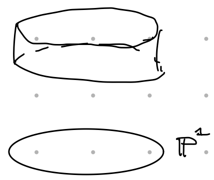
$$(x:y:z) \longmapsto (-x : iy : z)$$

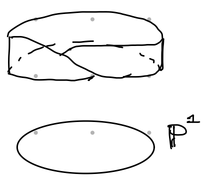
BUT THEY ARE NOT ISOMORPHIC OVER $\mathbb{Q}$

$$E_1[3](\mathbb{Q}) = 0, \qquad E_2[3](\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}.$$

(COUNTER) EG $\quad$ CONSIDER $E/\mathbb{C}$ ELLIPTIC CURVE, CONSTRUCT TWO FAMILIES $/ \mathbb{P}^1$

$E_{\mathrm{triv}} = E \times \mathbb{P}^1 \qquad\qquad E_{\mathrm{twis}}$ = GLUE TWO COPIES OF CONSTANT FAMILY ALONG $[-1]$ TWIST





FIBERWISE EC ARE INDISTINGUISHABLE $\quad E_{\mathrm{triv},t} \simeq E_{\mathrm{twist},t} \quad \forall\, t \in \mathbb{C}^*$

$\mathbb{P}^1 \longrightarrow X$ SENDS $\mathbb{P}^1$ TO SINGLE POINT

$\rightsquigarrow$ CANNOT DISTINGUISH TWO FAMILIES

MODULI PROBLEMS WITH EXTRA STRUCTURE

- A FULL LEVEL N-STRUCTURE FOR S SCHEME OVER $\operatorname{Spec}\mathbb{Z}[1/N]$ IS ISOM. OF LOCALLY FINITE GROUP SCHEMES

$$\alpha : (\mathbb{Z}/N\mathbb{Z})^2_S \simeq E[N]$$

DENOTE $[\Gamma(N)]$ FUNCTOR CLASSIFYING (E,α)

- DEFINE $[\Gamma_1(N)]$ AS SET OF MONOMORPHISMS OF FINITE ÉTALE GROUP SCHEMES

$$\alpha : (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow E[N]$$

- DEFINE $[\Gamma_0(N)]$ AS SET OF CLOSED SUBGROUP SCHEMES $H \subset E[N]$ ST $\forall\, \bar{s} \to S$ GEOMETRIC POINT, THE FIBER $H_{\bar{s}}$ IS CYCLIC OF ORDER N.

THEOREM
- IF $N > 3$, $[\Gamma_1(N)]$ IS REPRESENTABLE BY SMOOTH SCHEME $Y_1(N)$ AFFINE OF RELATIVE DIMENSION 1 OVER $\operatorname{Spec}\mathbb{Z}[1/N]$
- IF $N > 2$, $[\Gamma(N)]$ IS REPRESENTABLE BY SMOOTH SCHEME $Y(N)$ AFFINE OF RELATIVE DIMENSION 1 OVER $\operatorname{Spec}\mathbb{Z}[1/N]$

UNIVERSAL ELLIPTIC CURVES

LET $[\Gamma_1(N)]$ REPRESENTABLE MP

$$\underline{Sch}_{\mathbb{Z}[1/N]} \longrightarrow \underline{Sets}$$

$$S \longmapsto \{(E,\alpha) \mid E/S \text{ EC}, \alpha\ \Gamma_1(N)\text{-STRUCTURE}\}/\simeq_S$$

$$\simeq \operatorname{Hom}_{\underline{Sch}_{\mathbb{Z}[1/N]}}(S, Y_1(N)) = Y_1(N)(S)$$

$\rightsquigarrow$ SPECIAL ELEMENT $\quad id \in \operatorname{Hom}_{\underline{Sch}_{\mathbb{Z}[1/N]}}(Y_1(N), Y_1(N))$

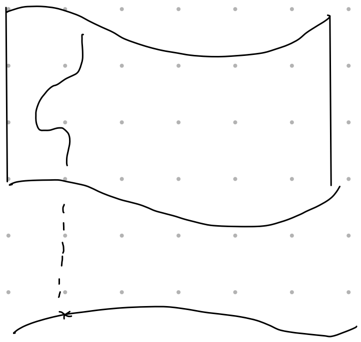
CORRESPONDING TO PAIR $(\mathcal{E}^{univ}, \alpha^{univ})$ $\quad \mathcal{E}^{univ}/Y_1(N)$ ELLIPTIC CURVE

FIX SCHEME S, LET $(E,\alpha) \in [\Gamma_1(N)](S)$, $\varphi \in Y_1(N)(S)$ ELEMENT.

WE HAVE CARTESIAN DIAGRAM

$$\begin{tikzcd}
(E,\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (\mathcal{E}^{univ}, \alpha^{univ}) \arrow[d] \\
S \arrow[r, "\varphi"'] & Y_1(N).
\end{tikzcd}$$

A concrete picture over $\mathbb{C}$



$$\begin{array}{c} (\mathcal{E}^{univ})^{an} \simeq \Gamma_1(N)\backslash \mathcal{H}\times\mathbb{C}/\mathbb{Z}^2 \\ \big\downarrow \\ Y_1(N)(\mathbb{C})^{an} \simeq \Gamma_1(N)\backslash\mathcal{H} \end{array}$$

$\mathbb{Z}^2 \curvearrowright \mathcal{H}\times\mathbb{C}$

$(m,n)\cdot(\tau,z) = (\tau, m+n\tau+z)$

An algebraic example

- $[\Gamma_1(4)]$ is representable by $Y_1(4) = \operatorname{Spec} \mathbb{Z}\left[\frac{1}{2}, d, (d(d-4))^{-1}\right]$

  The universal elliptic curve is given by

  $$dy^2z = x^3 + (d-2)x^2 + x$$

  The universal section of order 4 is given by

  $$P = (1:1:1)$$

The case $[\Gamma_0(p)]$

$[\Gamma_0(N)]$ classifies pairs (E,H) $H\subset E[N]$ locally finite flat cyclic subgroup

Automorphism $[-1]$ prevents representability

Consider $(\mathcal{E}^{univ}, P^{univ})$ universal EC $/\ Y_1(N)$

The pair $(\mathcal{E}^{univ}, d\,P^{univ})$ w/ $d\in(\mathbb{Z}/N\mathbb{Z})^\times$ defines morphism

$$\langle d\rangle : Y_1(N) \longrightarrow Y_1(N)$$

$$\rightsquigarrow (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \operatorname{Aut}(Y_1(N))$$

Consider the quotient scheme

$$Y_0(N) := G\backslash Y_1(N)$$

This defines the coarse moduli scheme for $[\Gamma_0(N)]$

$\rightsquigarrow$ $Y_0(N)(k)$ can be identified w/ equivalence classes (E,H) over k where two pairs are equivalent if they are isomorphic over algebraic closure of k.

$Y_1(N) \longrightarrow Y_0(N)$ finite flat but not necessarily étale

STUDY OF BAD REDUCTION $Y_0(p)$

CONSIDER $[\Gamma_1(N,p)]$, $p \nmid N$ MODULI PROBLEM OVER $\mathbb{Z}[1/N]$-SCHEMES CLASSIFYING (E, α, C) OVER S

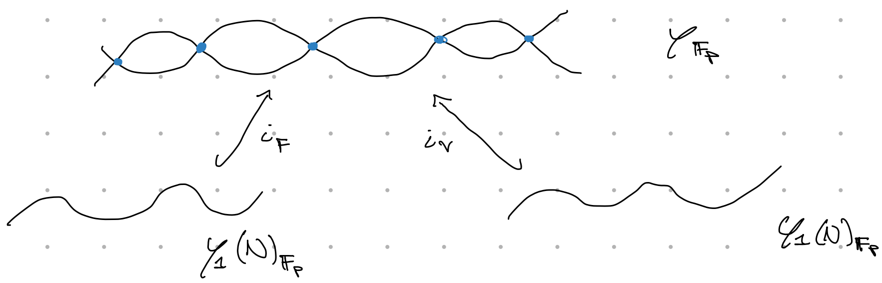
- E/S ELLIPTIC CURVE
- α MONOMORPHISM OF GROUP SCHEMES $\alpha : (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow E[N]$
- C FINITE FLAT SUBGROUP SCHEME W/ GEOM FIBERS OF RANK p

THEOREM (DELIGNE - RAPAPORT)

IF $N > 3$, THEN $[\Gamma_1(N,p)]$ IS REPRESENTABLE BY SCHEME $Y = Y_1(N,p)$ OVER $\mathbb{Z}[1/N]$. MOREOVER Y IS REGULAR AND $Y_{\mathbb{Z}[1/Np]}$ IS SMOOTH W/ IRREDUCIBLE GEOMETRIC FIBERS.

CONSIDER $Y_{\mathbb{F}_p} = Y \times \mathbb{F}_p$, $Y_1(N)_{\mathbb{F}_p} = Y_1(N) \times \mathbb{F}_p$, $\mathcal{E}_0 = (\mathcal{E}^{univ})_{\mathbb{F}_p}$, $\alpha_0 = (\alpha^{univ})_{\mathbb{F}_p}$

- FROBENIUS INDUCES p-ISOGENY $(\mathcal{E}_0, \alpha_0) \longrightarrow (\mathcal{E}_0^{(p)}, \alpha_0^{(p)})$
- VERSCHIEBUNG INDUCES p-ISOGENY $(\mathcal{E}_0^{(p)}, d\alpha_0^{(p)}) \longrightarrow (\mathcal{E}_0, \alpha_0)$ W/ $dp \equiv 1 \mod N$



§ 3 COMPACTIFICATION

OVER $\mathbb{C}$: $\Gamma \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \hookleftarrow Y(\Gamma)$

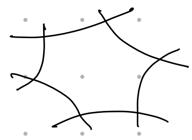
ADD POINTS TO COMPACTIFY MODULAR CURVE

DELIGNE - RAPAPORT INTRODUCED MODULI DESCRIPTION OF CUSPS

$\rightsquigarrow$ 'DEGENERATIONS OF ELLIPTIC CURVES'

def LET k ALGEBRAICALLY CLOSED FIELD.

THE NERON N-GON $(C_N, +)$ IS THE SCHEME COMING FROM GLUEING N COPIES OF $\mathbb{P}^1$



EXAMPLE OF C_6

$$v : \coprod_{i \in \mathbb{Z}/N\mathbb{Z}} \mathbb{P}^1 \longrightarrow C_N$$

WHERE $(\infty)_i$ AND $(0)_{i+1}$ ARE SEND TO THE SAME POINT

v RESTRICTS TO AN ISOMORPHISM ON THE SMOOTH LOCUS

$$v : \coprod_{i \in \mathbb{Z}/N\mathbb{Z}} \mathbb{G}_m \xrightarrow{\sim} C^{sm}$$

INDUCING A GROUP STRUCTURE AND A NATURAL ACTION

$$C^{sm} \times C \longrightarrow C$$
$$((x)_i, (y)_j) \longmapsto (xy)_{i+j}$$

def A GENERALIZED ELLIPTIC CURVE OVER S IS A PAIR $(E, +)$ W/

- E SCHEME OF CURVES $/S$
- $+$ IS S-MORPHISM $E^{sm} \times_S E \longrightarrow E$

AND WE REQUIRE THAT THE GEOMETRIC FIBERS OF $(E,+)$ ARE EITHER ELLIPTIC CURVES OR NÉRON POLYGONS

EG THE TATE CURVE OVER $S = \operatorname{Spec} \mathbb{Z}[[q]]$ IS DEFINED BY

$$E_q : Y^2Z + XYZ = X^3 + a_4 XZ^2 + a_6 Z^3 \quad \text{IN } \mathbb{P}^2_S$$

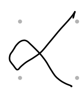
W/

$$a_4 = -5 \sum_{n \geq 1} n^3 q^n / (1-q^n), \qquad a_6 = -\frac{1}{12} \sum_{n \geq 1} (7n^5 + 5n^3) q^n / (1-q^n)$$

THEN E_q^{sm} IS THE COMPLEMENT OF CLOSED SUBSCHEME $X = Y = q = 0$

$s : \mathbb{Z}[[q]] \longrightarrow k$ DEFINING GEOMETRIC POINT $s(q) = 0$

$$E_{q,s} : Y^2Z + XYZ = X^3$$

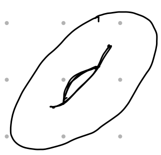


NERON 1-GON

IF $k = \mathbb{C}$, $s(q) = e^{2\pi i z}$ THEN $E_{q,s} \simeq \mathbb{C}^\times / e^{2\pi i z}$

$\uparrow \text{exp}$

$\mathbb{C}/\Lambda_z$



Consider the functor

$$\mathcal{G}_1(N) : \underline{Sch} \longrightarrow \underline{Set}$$

associating to a scheme S the set of isomorphism classes of pairs (E,P)

- E generalised elliptic curve over S
- P section $S \longrightarrow E^{sm}$ of exact order N

  and for every geometric point $s : \operatorname{Spec} k \longrightarrow S$ the image of resulting immersion $(\mathbb{Z}/N\mathbb{Z})_s \hookrightarrow E^{sm}_k$ meets every component

Theorem (Deligne–Rapoport)

If $N > 4$ then $\mathcal{G}_1(N)$ is representable by smooth curve $\mathcal{X}_1(N)$ over $\operatorname{Spec} \mathbb{Z}[1/N]$.

Can define analogous problem for $\Gamma_0(N)$

We then obtain a canonical map

$$\begin{tikzcd}
E_q \arrow[r] \arrow[d] & \mathcal{E}^{univ} \arrow[d] \\
\operatorname{Spec} \mathbb{Z}[1/N][[q]] \arrow[r, "i_q"'] & \mathcal{X}_1(N)
\end{tikzcd}$$

Think of q as local parameter at cusp ∞.

From this construction we obtain the q-expansion map

Consider $\Omega^1_{Y_1(N)/\mathbb{Z}[1/N]} \ni \omega_f$

$$\operatorname{Spec} \mathbb{Z}[1/N]((q)) \xrightarrow[i_q]{} Y_1(N) \quad \rightsquigarrow \quad i_q^* \omega_f = f(q) \cdot \frac{dq}{q} \qquad f(q) \in \mathbb{Z}[1/N][[q]]$$

§4. Hecke operators

Interpret Hecke operators in terms of correspondences

$$\begin{tikzcd}
 & Y_1(N,p) \arrow[dl] \arrow[dr] & \\
Y_1(N) & & Y_1(N)
\end{tikzcd} \qquad p \nmid N$$

$$\begin{tikzcd}
 & (E,\alpha,C) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(E,\alpha) & & (E/C, \alpha')
\end{tikzcd} \qquad \alpha' : (\mathbb{Z}/N\mathbb{Z})_S \hookrightarrow E/C\,[N]$$

Action on modular forms induced by pullback and pushforward under maps.