- MORPHISMS: $(U,V,\delta) \longrightarrow (U',V',\delta')$ COMMUTATIVE DIAGRAMS W/

  $U \hookrightarrow U'$ OPEN IMMERSION, $V \hookrightarrow V'$ MORPHISM COMP. W/ PD-STRUCTURES

- COVERINGS: AS FOR INFINITESIMAL SITE.

REM WE CAN DEFINE AND CHARACTERIZE SHEAVES AS IN INFINITESIMAL SITE

def CRYSTALLINE COHOMOLOGY IS DEFINED BY

$$H^i(X/W_n) = H^i((X/W_n)_{crys}, \mathcal{O}_{X/W_n}), \quad H^i(X/W) = \varprojlim H^i(X/W_n).$$

THEOREM IF Z/W_n IS A SMOOTH LIFT OF X, THE CRYSTALLINE COHOMOLOGY IS CANONICALLY ISOMORPHIC TO DE RHAM COHOMOLOGY

$$H^i(X/W_n) \simeq H^i_{dR}(Z/W_n)$$

CRYSTALLINE COHOMOLOGY II

§ REVIEW OF CRYSTALLINE SITE

LET k BE PERFECT FIELD, $W(k)$ RING OF WITT VECTORS

X BE A $W(k)$-SCHEME SUCH THAT p LOCALLY NILPOTENT (EG X/k)

THE CRYSTALLINE SITE $Crys(X/W)$ HAS

- OBJECTS: TRIPLES (U,T,δ), $U \subset X$ OPEN SUBSET

  $U \hookrightarrow T$ NILPOTENT THICKENING W/ DPS δ.

- MORPHISMS: COMMUTATIVE DIAGRAMS
- COVERINGS: INDUCED BY COVERINGS OF T.

THE STRUCTURE SHEAF $\mathcal{O}_{X/W}$ OF $Crys(X/W)$ IS DEFINED BY

$$\mathcal{O}_{X/W}((U,T,\delta)) = \Gamma(T, \mathcal{O}_T)$$

FOR EVERY B $W_n = W/p^nW$-ALGEBRA AND IDEAL J W/ DIVIDED POWERS δ, DEFINE THE MODULE OF DIFFERENTIALS COMPATIBLE WITH DIVIDED POWERS AS THE MODULE OF W-LINEAR DIFFERENTIALS QUOTIENTED BY RELATIONS

$$d(\delta_n(x)) = \delta_{n-1}(x)dx \quad \forall x \in J, \ n \geq 1$$

WRITE Ω_B FOR THIS MODULE

WE CAN DESCRIBE Ω_B FROM DIAGONAL EMBEDDING

GIVEN OBJECT (U,T,δ) OF $Crys(X/W)$ LET $(U, T(1), \delta(1))$ THE PRODUCT OF (U,T,δ) W/ ITSELF, SCHEME $T(1)$ IS DIVIDED POWER ENVELOPE OF

$U \hookrightarrow T \times_W T$, $\delta(1)$ INDUCED DPS.

DIAGONAL $\Delta: T \longrightarrow T(1)$ IS CLOSED IMMERSION CORR. TO SHEAF $\mathcal{I}$

WE THEN DEFINE THE SHEAF OF DIFFERENTIALS ON THE CRYSTALLINE SITE

$$\Omega^1_{X/W}((U,T,\delta)) = \Gamma(T, \mathcal{I}/\mathcal{I}^{[2]})$$

W/ $\mathcal{I}^{[2]}$ SECOND DIVIDED POWER OF $\mathcal{I}$.

DEFINE $\Omega^i_{X/W}$ AS i-TH EXTERIOR POWER OF $\Omega^1_{X/W}$.

PROP THERE IS A NATURAL MORPHISM OF TOPOI

$$u_{X/S} : (X/S)_{crys} \longrightarrow X_{zar}$$

GIVEN BY:

i) FOR $\mathcal{F} \in (X/S)_{crys}$ AND $j: U \hookrightarrow X$ OPEN IMMERSION

$$(u_{X/S*}\mathcal{F})(U) = \Gamma(U/S_{crys}, j^*_{crys}\mathcal{F})$$

ii) FOR $E \in X_{zar}$ AND $(U,T,\delta) \in Crys(X/S)$

$$(u_{X/S}^* E)(U,T,\delta) = \Gamma(U, E|_U)$$

WE HAVE A CANONICAL ISOMORPHISM

$$R\Gamma(X_{zar}, Ru_*\mathcal{F}) \simeq R\Gamma((X/W)_{crys}, \mathcal{F})$$

$$H^i(X_{zar}, Ru_*\mathcal{F}) \simeq H^i((X/W)_{crys}, \mathcal{F})$$

ASSUME X EMBEDS INTO Z SMOOTH OVER W, $j: X \hookrightarrow Z$ CLOSED IMMERSION.

IN GENERAL, IDEAL $Ker(\mathcal{O}_Z \to \mathcal{O}_X)$ DOES NOT HAVE DP STRUCTURE.

CONSIDER $\bar{Z}$ THE PD-ENVELOPE OF X IN Z, $X \hookrightarrow \bar{Z} \to Z$

FOR A CRYSTAL $\mathcal{E}$ THERE IS A UNIQUE INTEGRABLE CONNECTION

$$d: \mathcal{E}_{\bar{Z}} \longrightarrow \mathcal{E}_{\bar{Z}} \otimes \Omega^\bullet_{Z/S}$$

THEOREM THERE IS A CANONICAL ISOMORPHISM

$$Ru_{X/S*}\mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\bar{Z}} \otimes \Omega^\bullet_{Z/S}$$

$$H^\bullet_{crys}(X, \mathcal{E}) \xrightarrow{\sim} H^\bullet(X_{zar}, \mathcal{E}_{\bar{Z}} \otimes \Omega^\bullet_{Z/S})$$

COROL IF Z/W_n IS SMOOTH LIFT OF X, THEN $\bar{Z} = Z$ AND

$$H^\bullet_{crys}(X/W_n) \simeq H^\bullet_{dR}(Z/W_n).$$

REM FROBENIUS STRUCTURE.

LET $F: X \to X$ ABSOLUTE FROBENIUS

$\sigma: W_n \to W_n$ LIFT OF FROBENIUS MORPHISM.

BY FUNCTORIALITY OF CRYSTALLINE COH. $F: H^\bullet(X/W_n) \longrightarrow H^\bullet(X/W_n)$

IF X LIFTS TO Z/W_n THEN WE GET ACTION OF FROBENIUS ON $H^\bullet_{dR}(Z/W_n)$

EVEN IF IN GENERAL FROBENIUS DOES NOT LIFT TO Z.

§ WITT VECTORS, WITT RINGS, WITT SCHEME

THE GENERAL IDEA OF WITT VECTOR IS TO CONSTRUCT A LIFT OF A CHARACTERISTIC p OBJECT TO CHARACTERISTIC 0

LET $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[T_1, \dots, T_n]$ AFFINE SPACE

$\mathbb{A}^n$ REPRESENTS THE FUNCTOR $M \mapsto M^n$ FOR EVERY M RING

IT IS NATURALLY ENDOWED W/ RING STRUCTURE

$$\mathbb{A}^n(\mathbb{Z}) \ni (x_1, \dots, x_n), \quad (x_i)_i + (y_i)_i = (x_i + y_i)_i, \quad (x_i)_i \cdot (y_i)_i = (x_i \cdot y_i)_i$$

COMPONENT WISE

DENOTE E^n THE SCHEME IN RINGS $\mathbb{A}^n$. (SCHEME + RING FUNCTOR STRUCTURE)

THEN THERE EXISTS A UNIQUE RING SCHEME STRUCTURE ON $\mathbb{A}^n$ MAKING THE MORPHISM OF SCHEMES

$$(\varphi_i)_i : \mathbb{A}^n \longrightarrow E^n \qquad \text{'GHOST MAP'}$$

$$\varphi_i : (x_1, \dots, x_n) \longmapsto \sum_{j=1}^{i} p^{j-1} x_j^{p^{i-j}} = x_1^{p^{i-1}} + p x_2^{p^{i-2}} + \dots + p^{i-1} x_i$$

AN HOMOMORPHISM OF SCHEME OF RINGS

def W_n IS THE SCHEME $\mathbb{A}^n$ ENDOWED WITH THIS RING STRUCTURE CALLED WITT SCHEME

IN PARTICULAR, THERE EXIST $S_n, P_n \in \mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ SUCH THAT

$$\underline{a} + \underline{b} = (a_0 + b_0, S_1(\underline{a}, \underline{b}), \dots) \qquad \underline{a} \cdot \underline{b} = (a_0 b_0, P_1(\underline{a}, \underline{b}), \dots)$$

FOR EXAMPLE

- $S_1 = X_1 + Y_1 + \dfrac{X_0^p + Y_0^p - (X_0 + Y_0)^p}{p}$,
- $P_1 = X_0^p Y_1 + X_1 Y_0^p + p X_1 Y_1$

REM THE DEFINITION OF RING STRUCTURE COMES W/ UNIVERSALITY PROPERTY:

$f : X \longrightarrow W_n$ IS MORPHISM OF RING SCHEMES IFF $\forall\, 1 \leq i \leq n$ THE MAP $\varphi_i \circ f : X \longrightarrow E^n$ IS MORPHISM OF RING SCHEMES (RESP GROUP SCHEME)

- RESTRICTION HOMOMORPHISM OF RING SCHEMES

$$R_n : W_{n+1} \longrightarrow W_n$$

$$(x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, x_n)$$

- VERSCHIEBUNG HOMOMORPHISM OF GROUP SCHEMES (ADDITIVE)

$$V_n : W_n \longrightarrow W_n$$

$$(x_1, \dots, x_n) \longmapsto (0, x_1, \dots, x_{n-1})$$

- IN CHARACTERISTIC p, FROBENIUS HOMOMORPHISM OF RING SCHEMES

$$F_n : W_{n,\mathbb{F}_p} \longrightarrow W_{n,\mathbb{F}_p}$$
$$(x_1, \dots, x_n) \longmapsto (x_1^p, x_2^p, \dots, x_n^p)$$

EX VERIFY W/ UNIVERSAL PROPERTY THAT PREVIOUS MAPS ARE HOMOMORPHISMS

def THE LIMIT RING SCHEME OF WITT VECTORS IS GIVEN BY

$$W = \lim \left(\cdots W_{n+1} \xrightarrow[R_n]{} W_n \longrightarrow \cdots \right)$$

PROP IN CHARACTERISTIC p, ONE HAS THE RELATIONS ON $W_{\mathbb{F}_p}$:

$$F \circ V = V \circ F = [p]$$

PROOF SET

$$h = (h_1, h_2, \dots) = ([p] - VF)(x_1, x_2, \dots)$$

IT SUFFICES TO SHOW h_i IS DIVISIBLE BY p FOR ALL i.

WE WILL SHOW BY INDUCTION

$$\varphi_i(h) = p\varphi_i(x) - \varphi_i(VF(x)) = p^i x_i$$

- $i = 1$, $\quad h_1 = px_1, \quad \varphi_1(h) = px_1$
- $i = 2$,

$$\varphi_2(h) = h_1^p + ph_2 = h_1^p + p^2(x_2 - p^{p-2} x_1) = p^2 x_2$$

- $i > 2$ EX. $\square$

REM FROM THIS IT FOLLOWS $p = (0, 1, 0, \dots)$ IN $W_{\mathbb{F}_p}$.

def FOR ANY RING A THERE IS A NATURAL MAP

$$[-] : A \longrightarrow W(A)$$
$$a \longmapsto (a, 0, 0, 0, \dots)$$

CALLED TEICHMULLER MAP.

def LET X BE A SCHEME OVER $\operatorname{Spec} k$, k PERFECT FIELD OF char p.

DEFINE A SHEAF OF RINGS $W\mathcal{O}_X$ WHERE SECTIONS ON OPEN $U \subset X$

ARE GIVEN BY SEQUENCES

$$(a_1, a_2, a_3, \dots) \quad \text{w/ } a_i \in \Gamma(U, \mathcal{O}_X)$$

$$\Gamma(U, W\mathcal{O}_X) = W(\Gamma(U, \mathcal{O}_U))$$

WE CAN AS WELL DEFINE $W_n\mathcal{O}_X = W\mathcal{O}_X / V^n W\mathcal{O}_X$

$\rightsquigarrow W\mathcal{O}_X = \varprojlim_n W_n\mathcal{O}_X$

REM $(X, W\mathcal{O}_X)$ CAN BE THOUGHT AS A FORMAL LIFT TO CHARACTERISTIC 0 OF THE SCHEME $(X, \mathcal{O}_X)$.

REM FOR k PERFECT FIELD OF char p, THE k-POINTS OF $\mathbb{W}$ ARE OF PARTICULAR INTEREST

$$W = \mathbb{W}(k) = \lim_n \mathbb{W}_n = \lim_n \mathbb{W}_n(k)$$

IS A COMPLETE DISCRETE VALUATION RING W/ RESIDUE FIELD k AND UNIFORMIZER $p = (0, 1, 0, 0, \dots)$.

THERE IS AUGMENTATION ~~HOM.~~ GIVEN BY CANONICAL PROJECTION

$$W \longrightarrow W_1 = k.$$

THESE PROPERTIES CHARACTERIZE RING W UP TO UNIQUE ISOMORPHISM.

$K = W[1/p] = \operatorname{Frac} W$ IS OF CHARACTERISTIC 0.

§ DE RHAM - WITT COMPLEX

IF X SMOOTH k-SCHEME, ONE COULD NAIVELY TRY TO TAKE DE RHAM COMPLEX OF $\mathbb{W}(X)$ AND COMPUTE HYPERCOHOMOLOGY. $\rightsquigarrow$ DOES NOT WORK
- NOT FUNCTORIAL
- NOT COMPATIBLE W/ F AND V

ILLUSIE'S IDEA :
- EXTEND TO PROJECTIVE SYSTEM OF COMPLEXES
- EXTEND F AND V IN COMPATIBLE WAY.

- CLASSICAL PICTURE

def CALL ANTICOMMUTATIVE GRADED A-ALGEBRA B IN POSITIVE DEGREES W/ A-LINEAR DIFFERENTIAL

$$d : B^i \longrightarrow B^{i+1}$$

SUCH THAT $d^2 = 0$, $d(xy) = (dx)y + (-1)^i x(dy)$, A DIFFERENTIAL GRADED ALGEBRA

A MORPHISM OF DGA IS MORPHISM OF A-ALGEBRAS COMPATIBLE W/ THE DIFFERENTIAL STRUCTURES.

DENOTE $\underline{dga}^{\geq 0}(A)$ THE CATEGORY OF DGA.

PROP THE FUNCTOR

$$\begin{aligned} \underline{Alg}(A) &\longrightarrow \underline{dga}^{\geq 0}(A) \\ C &\longmapsto \Omega^{\bullet}_{C/A} \end{aligned}$$

IS LEFT ADJOINT TO THE FORGETFUL FUNCTOR

$$\begin{aligned} \underline{dga}^{\geq 0}(A) &\longrightarrow \underline{Alg}(A) \\ B &\longmapsto B^0 \end{aligned}$$

- ILLUSIE'S PICTURE

def A DE RHAM V-PROCOMPLEX IS A PROJECTIVE SYSTEM OF DGA'S ON X

$$M_\bullet = ((M_n)_{n \in \mathbb{Z}},\ R : M_{n+1} \longrightarrow M_n)$$

AND A FAMILY OF ADDITIVE MAPS

$$(V : M_n^i \longrightarrow M_{n+1}^i)_{n \in \mathbb{Z}} \qquad \text{ST} \qquad RV = VR$$

SATISFYING THE FOLLOWING PROPERTIES

- $M_{n\leq 0} = 0$, M_1 IS $\mathbb{F}_p$-ALGEBRA AND $M_n^0 = W_n(M_1^0)$ W/ R, V USUAL MAPS
- $\forall x \in M_n^i$, $y \in M_n^j$ $\quad V(x\,dy) = (Vx)\,dVy$
- $\forall x \in M_1^0$, $y \in M_n^0$ $\quad (Vy)\,d[x] = V([x]^{p-1}y)\,dV[x]$

A MORPHISM OF DE RHAM-V-PRO COMPLEXES IS A MORPHISM OF A PROJECTIVE SYSTEM OF DGA's COMPATIBLE W/ ADDITIONAL STRUCTURES.

DENOTE CATEGORY BY $\underline{VDR}(X)$.

<u>PROP</u> THE FORGETFUL FUNCTOR

$$\begin{aligned} \underline{VDR}(X) &\longrightarrow \underline{Alg}(X) \\ M_\bullet &\longmapsto M_1^0 \end{aligned}$$

HAS A LEFT ADJOINT

$$\begin{aligned} \underline{Alg}(X) &\longrightarrow \underline{VDR}(X) \\ A &\longmapsto W_\bullet \Omega_A \end{aligned}$$

CALLED DE RHAM-WITT COMPLEX.

FURTHERMORE, THE MORPHISM OF DGA's $\pi_n : \Omega^\bullet_{W_n A} \longrightarrow W_n \Omega^\bullet_A$ ST $\pi_n^0 = id$ IS SURJECTIVE AND $\pi : \Omega^\bullet_A \longrightarrow W_1 \Omega^\bullet_A$ IS AN ISOMORPHISM.

<u>PROP</u> THERE EXIST A UNIQUE ADDITIVE MORPHISM $F : W_n \Omega^\bullet_X \longrightarrow W_{n-1}\Omega^\bullet_X$ VERIFYING THE FOLLOWING PROPERTIES

$$F(ab) = F(a)F(b), \qquad FdV = d \text{ ON } W_n\mathcal{O}_X, \qquad F(d[x]) = [x]^{p-1}d[x]$$

AND SUCH THAT $F = R \circ \cancel{F}\text{rob}$ ON $W_n\mathcal{O}_X$

FURTHERMORE, WE HAVE THE FOLLOWING IDENTITIES

$$FV = VF = p, \qquad FdV = d, \qquad xVy = V(Fx\,y)$$

<u>THEOREM</u> (ILLUSIE)

LET X A k-SMOOTH SCHEME. THERE EXIST A CANONICAL ISOMORPHISM BETWEEN CRYSTALLINE COHOMOLOGY $H^\bullet(X/W_n)$ AND HYPERCOHOMOLOGY OF DE RHAM-WITT COMPLEX $H^\bullet(X, W_n\Omega^\bullet_X)$ COMPATIBLE W/ FROBENIUS ACTION ON CRYSTALLINE COHOMOLOGY AND DE RHAM WITT COMPLEX F

MORE GENERALLY, WE HAVE AN ISOMORPHISM OF DERIVED OBJECTS

$$Ru_*\mathcal{O}_{X/W_n} \xrightarrow{\sim} W_n\Omega^\bullet_X$$

IN CATEGORY $D(X/W_n)$ OF W_n-mod SHEAVES ON X.

THIS ISOMORPHISM IS FUNCTORIAL.