We want to define a cohomology theory s.t.

- cup products $H^i \times H^j \to H^{i+j}$
- Poincaré duality $H^i \times H^{2n-i} \to K(-n)$
- Lefschetz fixed point / trace formula
- comparison theorem: $X/\mathbb{C}K$ char $k=0$

$$H^i_{sing}(X) \otimes \mathbb{C} \cong H^i(X) \otimes_K \mathbb{C}$$

~~attempt 1: Let V/k a variety~~
~~$H^i_{sing}(V, \Lambda)$~~

attempt 2: $H^i(V, \underline{\Lambda})$ $\xrightarrow{\underline{\Lambda} \text{ constant}}$ also zero

if V is irreducible $\rightsquigarrow$ any open is connected

hence $\underline{\Lambda}$ is flasque

__Problem__: too few opens / opens are too big

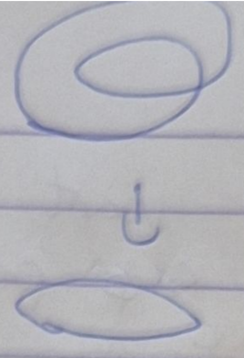
how to fix this?
- add more opens
- opens need not be subsets
  $U \to X$ not necessarily injective

Define a 'Grothendieck topology' on X which here the opens are étale maps $U \to X$
this will form a category $\left(\begin{array}{c} U \to U' \\ \searrow \ \swarrow \\ X \end{array} \right)$
which will replace __open(X)__.

étale maps are algebraic analogues of
- unramified covering maps of varieties $/\mathbb{C}$
- local diffeomorphisms of manifolds

ex $\quad \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$

$\qquad x \longmapsto x^n$



ex $\quad$ covering maps

Th if X is a variety $/\mathbb{C}$ then for any ~~étale~~ local diffeomorphism

$$(U,u) \longrightarrow (X,x)$$

there exists an ~~open~~ open $V \subset X$ such that

$$\begin{tikzcd} (V,x) \arrow[r, phantom, "\subseteq"] \arrow[dr] & (X,x) \\ & (U,u) \arrow[u] \end{tikzcd}$$

factors

which will give a map $H^i_{ét}(X,\Lambda) \to H^i_{sing}(X,\Lambda)$

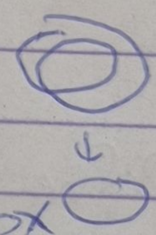
every "étale covering of x $(U,u) \to (X,x)$" can be refined to a ~~comp~~ covering in the complex topology.

let's try: local ~~diffeomorph~~ isomorphism? Zariski topology

$\mathbb{C}^\times \to \mathbb{C}^\times$

$z \mapsto z^n$

an isomorphism in no open of $\mathbb{C}^\times$

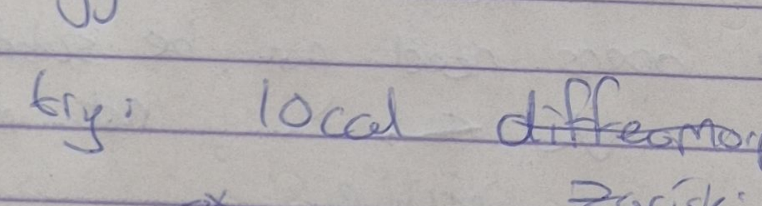


$f: X \to Y$

a local isomorphism of schemes induces isomorphisms on the local rings

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

"local rings see too much"



local rings are different

<u>but</u> the completions of the local rings are the same / tangent cones / tangent spaces

when X, Y are varieties over an algebraically closed field: we say $f: X \to Y$ is étale if the induced maps on the completed local rings are isomorphisms /
equivalently the maps on the tangent cones are iso
if X, Y non-singular: maps on the tangent space

$$\mathbb{C}^\times \longrightarrow \mathbb{C}^\times \qquad \left(\frac{dx^n}{dx}\right) = (nx^{n-1}) \neq 0$$
$$z \longmapsto z^n$$

"smooth of relative dimension 0"

def A map $X \xrightarrow{f} Y$ of schemes is étale if it is (locally) of finite type flat and unramified

Flat ↳ recall: $A \to B$ rings is flat if B is flat as A-module
$M \mapsto B \otimes_A M$ is exact. If A is I.D. then ~~if~~ ~~a~~ $A \to A$, $x \mapsto ax$ is injective

and hence $B \to B$, $x \mapsto ax$ is also injective

this implies $A \to B$ is injective
If is Dedekind domain, then $A \to B$ injective $\Rightarrow A \to B$ is flat

~~now~~ it suffices to check $A_{f^{-1}(m)} \to B_m$ is flat for all maximal ideals of B

A morphism of ~~sts~~ schemes $\varphi: X \to Y$ is flat if ~~$\mathcal{O}_{X,\varphi}$~~ $\mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ are flat

~~WLOG x~~ (suffices to check closed pts)

flat $\leftrightarrow$ continuous family of varieties paramterised by $x \in X$ $X_x \to \{x\}$

If φ is flat then
$$\dim Y_x = \dim Y - \dim X$$

moreover a ~~finite~~ finite morphism is flat ~~if~~ the fibers have the same amount of pts

unramified a local hom $f: A \to B$ is unramified if $B/f(m_A)B$ is a finite separable extension of A/m_A

$\iff$
- $f(m_A)B = m_B$
- B/m_B is finite separable over A/m_A

a morphism of schemes is unramified if it is finite type & the local homomorphism are unramified (suffices to check closed pts)

$\iff$ $f: X \to Y$ of finite type & $\Omega^1_{X/Y} = 0$

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ex $A \xrightarrow{f} B$ of rings is étale if $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale $\iff$
- B is f.g. A-alg
- ~~B is flat~~ f is flat
- for all maximal ideals $\mathfrak{n} \leq B$, $B_{\mathfrak{n}}/f(\mathfrak{p})B_{\mathfrak{n}}$ is finite separable over $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, $\mathfrak{p} = f^{-1}(\mathfrak{n})$

ex $f: Y \to X$, X, Y varieties $/k$ is étale $\iff$ ~~d.~~ ~~local~~ homomorphisms on the completed local rings are iso

$\Leftarrow$ easy: $\hat{A} \to \hat{B}$ is étale, then $A \to B$ is étale ex $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$ not étale (not algebraic)

ex $f: X \to \operatorname{spec} k$
since f is finite type, X is affine
since f is unramified, X has dimension 0
A is a unramified f.g k-algebra

$$\Downarrow$$

$$A = \prod_i l_i$$

of separable field extensions

ex $\mathbb{Q}$ ~~$\mathcal{O}_{K,S}$~~ K a number field
S-integers $\mathcal{O}_{K,S}$ then the étale coverings
are

$$\mathcal{O}_{L,S} \to \mathcal{O}_{K,S}$$

~~with $\mathcal{O}_L/\mathcal{O}$~~ with L/K separable unramified outside S
$\mathcal{O}_{L,S}$ the integral closure of ~~K~~ $\mathcal{O}_{K,S}$ in L
and $\mathcal{O}_{L,S}/\mathcal{O}_K$ unramified outside S

ex standard étale: if A ring $f \in A[t]$ monic
$b \in A[t]/(f)$ s.t. f' is invertible in
$A[t]/(f)[\frac{1}{b}]$ then

$$A \to A[t]/(f) \quad \text{is standard étale}$$

any étale map $Y \to X$ is ~~covered~~ locally
standard étale. (see notes)

other characterisation of étale maps:

locally of finite type +

$$\begin{tikzcd} \operatorname{Spec}(C/N) \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} C \arrow[r] \arrow[ur, dashed, "\exists !"] & X \end{tikzcd}$$

~~e.g.~~ $N^2 = 0$
e.g. $C = k[\varepsilon]/(\varepsilon^2)$

$\exists \geq 1$: formally smooth
$\exists \leq 1$: formally unramified
$\exists = 1$: formally étale

properties of étale maps

a) open immersions are étale
b) ϕ, ψ étale $\Rightarrow \phi \circ \psi$ étale
c) base changes remain étale
d) $\phi \circ \psi$ étale $\Rightarrow \psi$ étale

$$\begin{tikzcd} Y_1 \arrow[rr] \arrow[dr] & 2 & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

$Y_1 \to X$ étale $\Rightarrow$ $Y_2 \to X$ étale

$\varphi: Y \to X$ étale $\quad \varphi(y) = x$

prop a) $\dim \mathcal{O}_{Y,y} = \dim \mathcal{O}_{X,x}$
b) φ is quasi-finite
c) φ is open
d) X reduced $\Rightarrow$ Y reduced
e) X normal $\Rightarrow$ Y normal
f) X regular $\Rightarrow$ Y regular

Fundamental group

Recall: ~~a~~ a covering map $f: Y \to X$ is a map such that $\forall x \in X$ $\exists U$ nbhd of x s.t. $f^{-1}(U) \cong \bigsqcup U_i$ $U_i \cong U$

If X is "nice" (i.e. pathwise connected and semi-locally simply connected) then X has a universal cover $\tilde{X} \to X$ s.t. $\pi_1(\tilde{X}) = 1$

It is universal in the sense that

$$\begin{tikzcd} & (\tilde{X}, x) \arrow[dl, dashed, "\exists!"'] \arrow[d] \\ (Y, y) \arrow[r] & (X, x) \end{tikzcd}$$

Consider $Aut_X(\tilde{X}) = \{$ covering maps $\tilde{X} \to \tilde{X}$ covering $X\}$

there is a map $Aut_X(\tilde{X}) \longrightarrow \pi_1(X, x)$
$$g \longmapsto \pi(\text{ path from } \tilde{x} \to g\tilde{x})$$
conversely $\pi_1(X, x)$ acts on $\tilde{X}$, which gives the reverse map.

define $F: \underline{Cov(X)} \longrightarrow \underline{Set}$
$$(\pi: Y \to X) \longmapsto \pi^{-1}(x)$$
this functor is representable by $\tilde{X}$ ie
$$F(Y) \cong Hom_X(\tilde{X}, Y) \text{ functorially in } Y.$$

If we let $\pi_1(X, x) / Aut_X(\tilde{X})$ act on $Hom_X(\tilde{X}, Y)$ on the left this becomes an ~~auto~~ equivalence of categories $cov(X) \to \pi_1(X, x)$ sets with finitely many orbits

étale fundamental groups

~~finite~~ covering maps $\leftrightarrow$ finite étale maps

- $\rightarrow$ finite étale $\pi: Y \to X$ is closed & surjective
- $\rightarrow$ if X / variety / alg. closed, each fiber has the same amount of pts, and each $x \in X$ has a nbd $(U, u) \to (X, x)$ s.t.
  $Y \times_X U$ is disjoint union of ~~cl.~~ open subvarieties mapped isomorphically onto U by $\pi \times 1$

~~Def~~ Consider $\underline{Fet/X}$ the category of finite étale maps $\to X$. Choose a geometric pt
$\bar{x}: Spec(k) \to X$, k separably closed field

Define $F: \underline{Fet/X} \longrightarrow$ Sets sending
$(\pi: Y \to X) \to Hom_X(\bar{x}, Y)$
"set of $\bar{x}$ valued pts of Y lying over x

if X is a variety over an algebraically closed field then $Hom_X(\bar{x}, Y) = \pi^{-1}(x)$

We cannot define a universal covering

ex there is no biggest covering of $\mathbb{A}^1 - \{0\}$
to $\mathbb{A}^1 - \{0\} \longrightarrow \mathbb{A}^1 - \{0\}$
$x \longmapsto x^n$
"there is no biggest" $\rightsquigarrow$ $(exp: \mathbb{C} \to \mathbb{C} - \{0\})$

$\rightsquigarrow$ F is not representable

So we will take limits

There is a projective system
$\tilde{X} = (X_i)_{i \in I}$ with I a directed set
such that

$$F(Y) = \mathrm{Hom}(\tilde{X}, Y) := \varinjlim_{i \in I} \mathrm{Hom}(X_i, Y)$$

Functorially in Y $\rightsquigarrow$ F is pro-representable

We call $\tilde{X}$ the universal covering space of X if X_i has degree over X equal to the order of $\mathrm{Aut}_X(X_i)$. $\rightsquigarrow$ X_i is Galois over X

If $\underset{I}{i} \leq \underset{I}{j}$ there is a map $X_j \to X_i$ induces a homomorphism $\mathrm{Aut}_X(X_j) \to \mathrm{Aut}_X(X_i)$, so define

$$\pi_1(X, \tilde{x}) = \mathrm{Aut}_X(\tilde{x}) := \varprojlim_i \mathrm{Aut}_X(X_i)$$

ex $X_n = \longrightarrow \mathbb{A}^1 - \{0\}$

$\mathbb{A}^1 - \{0\} = X_n \to \mathbb{A}^1 - \{0\}$

$t \longmapsto t^n$

then $\mathrm{Aut}_{\mathbb{A}^1 - \{0\}}(X_n) = \mu_n(k)$

thus $\pi_1(\mathbb{A}^1 - \{0\}, \tilde{x}) = \varprojlim_n \mu_n(k) \cong \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$

Thm $Y \mapsto F(Y)$ defines an equivalence from category of finite étale coverings of X to category of finite discret $\pi_1(X, \tilde{x})$-sets.

variety over $\mathbb{C}$ $\rightsquigarrow$ Riemann existence theorem
finite étale coverings $\to$ finite covering spaces
is an equivalence of categories

$$\pi_1(X, x) = \varprojlim_i \mathrm{Aut}_X(X_i) = \varprojlim \mathrm{Aut}_{X(\mathbb{C})}(X_i(\mathbb{C}))$$

$$= \widehat{\pi_1(X(\mathbb{C}), x)}$$

ex $\pi_1(\mathbb{C} - \{0\}) = \mathbb{Z}$