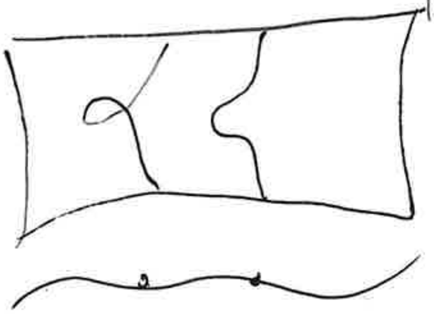
Etale topology and cohomology.

How to get "right" cohomology?

<u>IDEA</u>:

1. $H^1(X) = \pi^1(X)^{ab} \rightsquigarrow$ coverings of X explain right H^1

2. "Dévissage" (unwindings)



$$\begin{array}{c} Y \\ \downarrow \\ X \end{array}$$ fibration in curves

Coh of fibers + Coh of X $\xrightarrow{\text{spectral seq.}}$ Coh of Y

Good $H^1 \rightsquigarrow$ Good H^n by dim. induction

<u>Last time</u>: (All schemes are locally Noeth.)

$f: Y \to X$ is <u>étale</u> if it's flat and unram.

$\iff$ locally $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$

$B = A[T_1 \ldots T_n]/(P_1 \ldots P_n)$ where $\det\left(\frac{\partial P_i}{\partial T_j}\right) \in B^\times$.

X f.t. $/ \mathbb{C} \rightsquigarrow$ complex analitic space X^{an}

f etale $\overset{\text{"implicit function th"}}{\iff} f^{an}: Y^{an} \to X^{an}$ is a local homeom.

f finite étale $\iff f^{an}$ is a covering

new "opens" of X

Zariski-opens $U \subset X$ $\Rightarrow$ étale sch. $/X$ $\Leftarrow$ coverings $\mathcal{E} \downarrow \mathcal{O}$, $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$

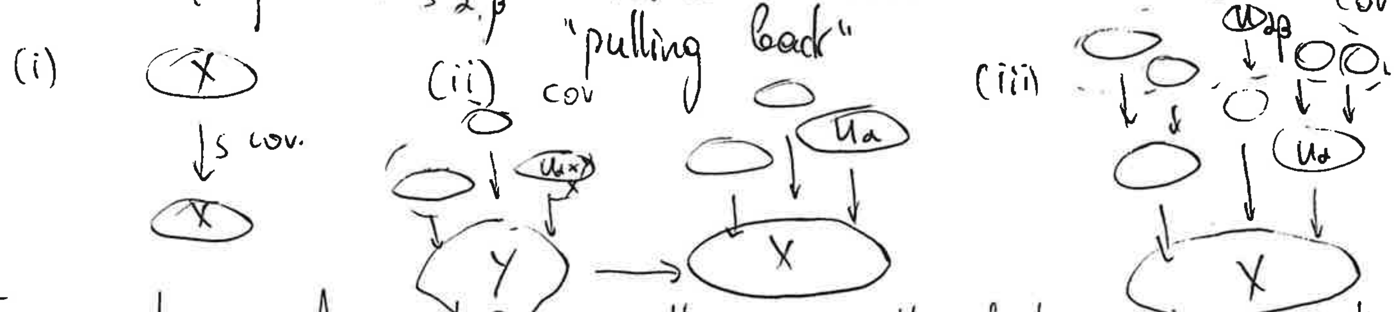
$\mathcal{C}$ category with fiber products — different terminology in SGA4

Def. A Grothendieck topology on $\mathcal{C}$ is given by a set of families of morphisms $\{U_\alpha \to X\}_\alpha$ ("coverings") for all $X \in \mathcal{C}$ s.t.

(i) iso. $U \xrightarrow{\sim} X$ is a cov.

(ii) $\forall$ cov. $\{U_\alpha \to X\}$ & $\forall$ morphism $Y \to X$: fiber products $\{U_\alpha \times_X Y \to Y\}_\alpha$ ~~exists and~~ form a cov.

(iii) $\forall$ cov. $\{U_\alpha \to X\}_\alpha$ if we are given a cov. $\{U_{\alpha\beta} \to U_\alpha\}_\beta$ $\forall \alpha$, the compositions $\{U_{\alpha\beta} \to X\}_{\alpha,\beta}$ form a cov. "refinement"



Examples. A category with a Grothendieck topology is called site.

(i) trivial (only cov. $X \xrightarrow{\sim} X$)

(ii) discrete (any family of morph. is a cov.)

(iii) $\mathcal{C} = \mathrm{Top}$, coverings = actual coverings open embeddings $U_\alpha \hookrightarrow X$ s.t. $X = \bigcup \mathrm{im}(U_\alpha)$

(iii)' X top. space, $\mathcal{C} = \mathrm{Open}(X)$, same cov.

(iv) "Zariski site"

X scheme

Small (resp. big) étale site X_{et} is

$\mathcal{C} = \{$étale schemes of f.t. $/X\}$ (resp. $\{$schemes of f.t. $/X\}$)

$\{U_\alpha \to V\}$ is a covering if (i) $U_\alpha \to V$ are étale of f.t.
(ii) $V = \bigcup \mathrm{im}(U_\alpha)$

X_{fl} big flat site

__Def.__ A presheaf of sets (resp. abelian groups) on a category $\mathcal{C}$ is a functor $\mathcal{F}: \mathcal{C} \to \mathrm{Set}^{op}$ (resp. $\mathcal{F}: \mathcal{C} \to \mathrm{Ab}^{op}$)

~~Assume $\mathcal{C}$ is a site.~~

$$\mathrm{PSh}_{set}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathrm{Set}^{op})$$

Assume $\mathcal{C}$ is a site.

$\mathcal{F} \in \mathrm{PSh}(\mathcal{C})$ is a __sheaf__ if $\forall V \in \mathcal{C}$ $\forall$ cov. $\{U_\alpha \to V\}$

$$\begin{tikzcd}[column sep=small] & U_\alpha \times_V U_\beta \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ U_\alpha \arrow[dr] & & U_\beta \arrow[dl] \\ & V & \end{tikzcd}$$

$$\mathcal{F}(V) \hookrightarrow \prod_\alpha \mathcal{F}(U_\alpha) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{\alpha,\beta} \mathcal{F}(U_{\alpha\beta})$$

is an equalizer of sets.
($\mathcal{F}(V)$ is sent injectively to the subset where $p_1^* = p_2^*$)

in étale topology can check for one $U \to V$, $U = \bigsqcup U_\alpha$

$\mathrm{Sh}_{set}(\mathcal{C}) \subset \mathrm{PSh}_{set}(\mathcal{C})$ is a full subcat.

__Example.__ $\mathcal{C} = \mathrm{Open}(X) \rightsquigarrow$ usual sheaves

__Theorem.__ The inclusion $\mathrm{Sh}(\mathcal{C}) \subset \mathrm{PSh}_{set}(\mathcal{C})$ admits a left adjoint ${}^{*}: \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$.

~~Idea of proof.~~ $\left[\mathrm{Hom}_{Sh}(\mathcal{F}^*, \mathcal{G}) = \mathrm{Hom}_{PSh}(\mathcal{F}, \mathcal{G}) \quad \forall \mathcal{G} \in \mathrm{Sh}(\mathcal{C})\right]$

~~Idea of proof~~ __Construction:__ "zero ~~Che~~ Čech cohomology"

$$\mathcal{F} \in \mathrm{PSh}(\mathcal{C}) \rightsquigarrow \check{\mathcal{H}}^0 \mathcal{F}(V) = \varinjlim_{\{U_\alpha \to V\}} \mathrm{eq}\left(\prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_{\alpha\beta})\right)$$

$$\mathcal{F}^* = \check{\mathcal{H}}^0(\check{\mathcal{H}}^0 \mathcal{F})$$

__Example.__ M abelian group, ~~$\mathcal{C}$ site~~

$\rightsquigarrow$ constant presheaf $\underline{M}(V) = M \quad \forall V \in X_{et}$

$\rightsquigarrow$ constant sheaf $\underline{M} = M^*$ ~~$\underline{M}(V) = M^{\#\{\text{connected components of } V\}}$~~

__Ex.__ If X is Noetherian,

$$\underline{M}(V) = M^{\#\{\text{connected components of } V\}} \quad \forall V \in X_{et}.$$

Prop. Let $\mathcal{C}$ be a site.
Then $Sh_{ab}(\mathcal{C})$ is an abelian category with enough injectives.

Proof.

$$\ker(\mathcal{F} \to \mathcal{G})(V) = \ker(\mathcal{F}(V) \to \mathcal{G}(V))$$

$$\operatorname{coker}_{psh}(\mathcal{F} \to \mathcal{G})(V) = \operatorname{coker}(\mathcal{F}(V) \to \mathcal{G}(V))$$

$$\operatorname{coker}(\mathcal{F} \to \mathcal{G}) = \left(\operatorname{coker}_{psh}(\mathcal{F} \to \mathcal{G})\right)^{\#}$$

enough injectives follows from general nonsense (Grothendieck's Tôhoku)

$$\left.\begin{array}{l} \exists \text{ coproducts} \\ + \\ \text{~~the limits~~ } \exists \text{ direct limits which are exact} \\ + \\ \exists \text{ generator} \end{array}\right\} \Longrightarrow \text{enough injectives}$$

□

$X \in \mathcal{C}$ final object

$$\Gamma : Sh_{ab}(\mathcal{C}) \longrightarrow \mathcal{A}b$$
$$\mathcal{F} \longmapsto \mathcal{F}(X)$$

$$H^n(\mathcal{C}, \mathcal{F}) := R^n\Gamma(\mathcal{F})$$

$X \in X_{ét}$ is final

X scheme
$\mathcal{F}$ sheaf of ab. gr on $X_{ét}$

$$\boxed{H^n_{ét}(X, \mathcal{F}) := H^n(X_{ét}, \mathcal{F})}$$

Ex. X top space $\rightsquigarrow$ $LH(X) = \{\text{local homeomorphisms } Y \to X\}$

covers: $\{U_\alpha \to V\}$ loc hom. s.t. $V = \bigcup \operatorname{im}(U_\alpha)$

(i) Check that LH is a site
(ii) Show that $Sh(LH(X)) \xrightarrow{\sim} Sh(X)$ is an equiv.
(iii) Deduce that $H^n(LH(X), \mathcal{F}) \xleftarrow{\sim} H^n(X, \mathcal{F})$ is an iso.

Hint: local homeom. is majorized by an open cov. $\bigsqcup U_\alpha \twoheadrightarrow Y \to X$

Ex. $X = \operatorname{Spec} k$, k is a field.

(i) Any étale $Y \to X$ of f.t. has form $Y = \coprod \operatorname{Spec} L_i$ where $K \subset L_i$ finite separable.

(ii) $Sh_{ab}(X_{ét}) \xrightarrow[\sim]{\text{equiv.}} \Gamma_K$ - modules $\quad \Gamma_K = Gal(K^{sep}/K)$

$\mathcal{F} \longmapsto \mathcal{F}(\operatorname{Spec} k^{sep}) = \overline{\mathcal{F}}$ $\circlearrowleft \Gamma_K$

(iii) Hence, $H^n(\operatorname{Spec} K, \mathcal{F}) \xleftarrow{\sim} H^n(\Gamma_K, \overline{\mathcal{F}})$
Galois cohomology

Example $X = \operatorname{Spec} K \quad K = K^{sep}$

- Any étale cover has form $\coprod X \to X$.

Hence, $Sh_{ab}(X) \xrightarrow{\sim} Ab$ and $H^n(X, \mathcal{F}) = 0 \ \forall \mathcal{F}$ for $n > 0$.

$\mathcal{F} \mapsto \Gamma(\mathcal{F})$

Functoriality

$X \xrightarrow{f} Y$ morphism of schemes

$\mathcal{F} \in Sh(X_{et}) \rightsquigarrow f_*\mathcal{F} \in Sh(Y_{et})$

$$f_*\mathcal{F}(U) := \mathcal{F}(U \times_Y X)$$

is a sheaf

$$\boxed{\begin{tikzcd} U \times_Y X \arrow[r] \arrow[d] & U \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}}$$

$\mathcal{G} \in Sh(Y_{et}) \rightsquigarrow f^*\mathcal{G} \in Sh(X_{et})$

$$f^p\mathcal{G}(V) = \varinjlim_{U \text{ as in } (*)} \mathcal{G}(U)$$

$$\boxed{\begin{tikzcd} (*)\, V \arrow[r] \arrow[d] & U \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}}$$

Prop. $f_* : Sh(X_{et}) \to Sh(Y_{et})$ admits an exact left adjoint $f^* : Sh(Y_{et}) \to Sh(X_{et})$

Proof

$\cancel{PSh(X_{et}) \underset{f_p}{\overset{f^p}{\rightleftarrows}} PSh(Y_{et})}$

$$f^*\mathcal{G} = (f^p\mathcal{G})^\#$$

$$Hom_{sh}(f^*\mathcal{G}, \mathcal{F}) = Hom_{psh}(f^p\mathcal{G}, \mathcal{F}) = Hom_{psh}(\mathcal{G}, f_*\mathcal{F}) \quad \square$$

$$Sh_{ab}(X_{et}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} Sh_{ab}(Y_{et})$$

~~Example~~ $\cancel{K = \bar{K}^{sep}, \; X \xrightarrow{\pi} \operatorname{Spec} K}$

$\cancel{\mathcal{F} \in Sh \quad \pi_* \quad Sh_{ab}(X_{et})}$

f_* is left exact because it has a left adjoint

f^* is exact

(~~righ~~ left ex. by adj. ~~f^p is~~ left exact on left exact since both ~~f^p and $\#$ are~~ It's a colimit)

$\cancel{Rf} \rightsquigarrow$ derived pushforwards

$$R^n f_* : Sh(X_{et}) \to Sh(Y_{et})$$

Exemple

(i) $K = K^{sep}$ $\quad X \xrightarrow{f} \operatorname{Spec} K$

$\mathcal{F} \in Sh_{ab}(X) \quad f_* \mathcal{F} = \underline{\Gamma(X, \mathcal{F})}$

$R^n f_* \mathcal{F} = H^n_{ét}(X, \mathcal{F})$.

(ii) $U \xrightarrow{f} X$ étale

$f^* \mathcal{F} = \mathcal{F}|_U$ restriction $\begin{tikzcd} V \arrow[r, "ét"] \arrow[rr, bend right] & U \arrow[r, "ét"] & X \end{tikzcd}$

$f^* \mathcal{F}(V) = \mathcal{F}(V)$

Faithfully flat descent

Slogan: flat morphisms localize well many objects/properties in AG.

$X = \bigcup U_\alpha$ open cover $\quad$ $\rightsquigarrow$ uniquely defines $X \to T$

$+\ f_\alpha : U_\alpha \to T$

s.t. $f_\alpha|_{U_{\alpha\beta}} = f_\beta|_{U_{\alpha\beta}}$ $\forall \alpha, \beta$

$\Longleftrightarrow$ $\mathrm{Hom}(-, X)$ is a sheaf in Zariski topology.

Sch_{fpqc} = ~~sets~~ all schemes with covers given by $\{U_\alpha \to X\}$ where $U_\alpha \to X$ flat q.-comp.

$X = \bigcup \mathrm{im}(U_\alpha)$

<u>Theorem</u>. ("descent for morphisms")
Let T be a scheme. Then $\mathrm{Hom}(-, T)$ is a sheaf of sets on Sch_{fpqc} (and hence on X_{et} $\forall X$)

<u>Proof</u>. We need to check the sheaf condition for all covers $\{U_\alpha \to X\}$

<u>Step 1</u>. Take $U = \coprod U_\alpha$. May assume there is only one U.

$$\mathrm{Hom}(X, T) \longrightarrow \mathrm{Hom}(U, T) \rightrightarrows \mathrm{Hom}(U \times_S U, T)$$

$$(*) \quad \begin{tikzcd} U \times_S U \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \arrow[dr] & \\ U \arrow[r, "f"] \arrow[d] & T \\ X \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

For any faithfully (surj.) flat q.-comp. $U \to X$, a morphisms $f : U \to T$ s.t. ~~compositions~~ $f \circ p_1 = f \circ p_2 : U \times U \to T$ factors uniquely through X.

<u>Step 2</u> Reduce $(*)$ to the case X, U, T affine.

<u>Step 3</u>. $X = \mathrm{Spec}\, A$ $\quad U = \mathrm{Spec}\, B$ $\quad T = \mathrm{Spec}\, C$

<u>Lemma</u>. $A \to B$ faith. flat ring map.

Then $0 \to A \to B \xrightarrow{b \mapsto b \otimes 1 - b \otimes 1} B \otimes_A B$ is exact seq. of A-modules

$\Rightarrow 0 \to \mathrm{Hom}_{gr}(C,A) \to \mathrm{Hom}_{gr}(C,B) \to \mathrm{Hom}_{gr}(C, B\otimes_A B)$ is exact.

$\Rightarrow \quad \mathrm{Hom}_{rg}(C,A) \to \mathrm{Hom}_{rg}(C,B) \rightrightarrows \mathrm{Hom}_{rg}(C, B\otimes_A B)$ in an eq. $\square$

~~Example. Obs.~~

<u>Example</u> X scheme. Any group scheme defines a sheaf on $X_{ét}$.

(i) $\mathbb{G}_a(V) = \Gamma(V, \mathcal{O}_V)$ is a sheaf on $X_{ét}$ since it's represented by $\mathbb{G}_a = \mathrm{Spec}\, \mathbb{Z}[T]$

(ii) $\mathbb{G}_m(V) = \Gamma(V, \mathcal{O}_V^\times)$ on $X_{ét}$

$\mathbb{G}_m = \mathrm{Spec}\, \mathbb{Z}[T, T^{-1}]$

(iii) $\mu_n(V) = \{x \in \Gamma(V, \mathcal{O}_V^\times) \mid x^n = 1\}$

$\mu_n = \mathrm{Spec}\, \mathbb{Z}[T]/(T^n - 1)$

<u>Prop.</u> ("Kummer sequence") Let $n \in \mathcal{O}_X^\times$.
Then $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m \to 1$ is exact on $X_{ét}$.

<u>Proof.</u> $1 \to \mu_n(V) \to \mathbb{G}_m(V) \to \mathbb{G}_m(V)$ is exact $\forall V$

$\rightsquigarrow \quad 1 \to \mu_n \to \mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m$ is exact on $X_{ét}$

Remains: surjectivity of $\mathbb{G}_m \xrightarrow{[n]} \mathbb{G}_m$.

$s \in \mathbb{G}_m(V) = \Gamma(V, \mathcal{O}_V^\times)$

We need an étale cover $U \xrightarrow{\pi} V$ s.t. $\pi^* s \in \Gamma(U, \mathcal{O}_U^\times)$ is an n-th power.

$U = \mathrm{Spec}\, \mathcal{O}_V[T]/(T^n - s)$

$\frac{\partial}{\partial T}(T^n - s) = n T^{n-1}$ is invertible in $\mathcal{O}_V[T]/(T^n - s)$:

$$n T^{n-1} \cdot \frac{T}{sn} = \frac{T^n}{s} = \frac{s}{s} = 1. \quad \square$$