Lecture 4: Cohomology of Curves.

Last time:

- (Small) étale site of a scheme X + sheaves on X_{et}
- Definition of étale cohomology: $H^i_{et}(X,\mathcal{F}) = R^i\Gamma_X(\mathcal{F})$ (examples were constant sheaves)
- 'Calculation' of $H^i_{et}(X,\mathcal{F})$ in the case that $X = \mathrm{Spec}(k)$ for k a field: $H^i_{et}(\mathrm{Spec}(k),\mathcal{F}) = H^i_{\text{Gr-gp}}(\Gamma_k, \mathcal{F}_{\bar{x}})$, $\mathcal{F}_{\bar{x}} = \varinjlim_{L/k \text{ galois}} \mathcal{F}(L)$.
- Functors $(\underset{\text{adjoints}}{f^* , f_*})$ $f_* \rightsquigarrow R^i f_*$

This time:

(1) Basic case of faithfully flat descent will give us 'more sheaves'

(2) Leray spectral sequence associated to morphism $f: X \to Y$.

(3) Try to calculate $H^i_{et}(C, \mathbb{Z}/n)$ for $C/\bar{k}$ a smooth curve.

Part 1: more sheaves.

Def: Let X be a scheme. An X-group-scheme G over X is a morphism $G \to X$ together with X-maps:

$\mu: G \times_X G \to G$, $\quad inv: G \to G$, $\quad e: X \to G$

that satisfy the 'group axioms': There are commuting diagrams:

$$\begin{tikzcd} G \times_X G \times_X G \arrow[r, "id \times \mu"] \arrow[d, "\mu \times Id"'] & G \times_X G \arrow[d, "\mu"] \\ G \times_X G \arrow[r, "\mu"] & G \end{tikzcd} \qquad \begin{tikzcd} X \times_X G \arrow[r, "e \times Id"] \arrow[dr, "p_2", "\sim"'] & G \times_X G \arrow[d, "\mu"] & G \times_X X \arrow[l, "Id \times e"'] \arrow[dl, "\sim"', "p_1"] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "(inv, Id)"] \arrow[d] & G \times_X G \arrow[d, "\mu"] & G \arrow[l, "(Id, inv)"'] \arrow[d] \\ X \arrow[r, "e"] & G & X \arrow[l, "e"'] \end{tikzcd}$$

Skip?

Remark: Equivalently a ~~group~~ scheme over X, $G \to X$, is an X-group scheme if $\mathrm{Hom}_X(-, G): Sch/X \to Set$ factors via $\mathrm{Forget}: Grp \to Set$.

X a scheme,

<u>Examples</u>: - For G a group, $\coprod_{g \in G} X_g$ is a constant group scheme.

- $\mu_{n,\mathbb{Z}} := \operatorname{Spec}(\mathbb{Z}[x]/(x^n - 1)) \overset{\text{group scheme}}{\leadsto} \mu_n \times_{\operatorname{Sp}(\mathbb{Z})} X$

functor of points $Y \mapsto \mu_n(\mathcal{O}(Y))$.

- $G_{m,\mathbb{Z}} := \operatorname{Spec}(\mathbb{Z}[x, x^{-1}]) \overset{\text{group scheme}}{\leadsto} G_m \times_{\operatorname{Sp}(\mathbb{Z})} X$.

functor of points is $Y \mapsto \mathcal{O}(Y)^{\times}$.

~~μ_n, G_m can also be defined for non-affine X by glueing.~~

We want these group schemes to give us sheaves on $X_{\acute{e}t}$.
Recall that a presheaf $\mathcal{F}$ is a sheaf if for any covering $(U_i \to V)_{i \in I}$ the follow sequence is exact in Set:

$$(*): \quad \mathcal{F}(V) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{(k,j) \in I^2} \mathcal{F}(U_k \times_V U_j) \text{ is exact.}$$

<u>Lemma 1</u>: A presheaf $\mathcal{F}$ on $X_{\acute{e}t}$ is a sheaf if and only if: (a) $\mathcal{F}$ satisfies $(*)$ for all Zariski-open cover $(U_i \to V)_{i \in I}$.
(b) $\mathcal{F}$ satisfies $(*)$ for any cover $(U \to V)$ with U, V affine.

<u>Sketch of proof</u>:
(a) implies that we have $\mathcal{F}(\coprod_{i \in I} U_i) = \prod_{i \in I} \mathcal{F}(U_i)$.
Combining this with (b) shows that $(*)$ holds for any $(U_i \to U)_{1 \le i \le n}$ with all U_i affine (use a 9-term comm. diagram and the fact that $\coprod_{1 \le i \le n} U_i$ is affine).
A diagram chasing argument as in [Milne's book II. Prop 1.5], shows that $(*)$ holds for any $(U_i \to U)_{i \in I}$. $(\square)$

The following lemma is a very basic form of what is called 'faithfully flat descent'

Lemma 2: Let $R \to A$ be a faithfully flat ring map. The complex of R-modules: $0 \to R \to A \xrightarrow[-1\otimes Id]{Id\otimes 1} A \otimes_R A$ is exact.
Proof: In the exercise sheet, hints provided (☒).

Now we can prove the following proposition giving us more sheaves.

Proposition 1: Let $Y \to X$ be an affine ~~scheme~~ morphism (resp. group-scheme) (resp. commutative group scheme). Then the functor $Hom_X(-,Y): X_{et} \to Set$ (resp. $X_{et} \to Grp$) (resp. $X_{et} \to Ab$) is a sheaf.
Proof: We use the criterion of lemma 1. We know that the sheaf-property (*) holds for Zariski covers by 'gluing morphisms'.
So it suffices to show that for $(U \to V)$ a cover with $V = Spec(R)$, $U = Spec(A)$, the sequence $Hom_X(V,Y) \to Hom_X(U,Y) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} Hom_X(U \times_V U, Y)$ (**) is exact in Set. Set $Y_V = Y \times_X V$, which is affine, $Y_V = Spec(B)$.
~~(**)~~ (**) can be rewritten as $Hom_V(V, Y_V) \to Hom_V(U, Y_V) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} Hom_V(U \times_V U, Y_V)$.
Since Y_V is affine this last sequence is identified with:

$$Hom_{R\text{-}Alg}(B,R) \to Hom_{R\text{-}Alg}(B,A) \overset{(Id\otimes 1)_*}{\underset{(1\otimes Id)_*}{\rightrightarrows}} Hom_{R\text{-}Alg}(B, A \otimes_R A) \quad (***)$$

By Lemma 2 applied to $R \to A$ (faithfully flat since $U \to V$ is surjective), we see that $R \to A$ is the equalizer of $A \overset{Id\otimes 1}{\underset{1\otimes Id}{\rightrightarrows}} A \otimes_R A$.
Since $Hom_{R\text{-}Alg}(B,-)$ preserves finite limits, we conclude that (***) is exact. □.

Remark: The proposition also holds for $Y \to X$ not nec. affine, see [Milne's book Theorem II.2.17]

In $Sh(X_{et}) := Sh_{Ab}(X_{et})$ we can talk about exact sequences.
Property of higher derived functors:

$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact

$\Rightarrow 0 \to R^0F(\mathcal{F}) \to R^0F(\mathcal{G}) \to R^0F(\mathcal{H})$

$\to R^1F(\mathcal{F}) \to R^1F(\mathcal{G}) \to \dots$ exact.

(Kummer-Sequence)

<u>Lemma 3</u>: Let X be a scheme and set $n \in \mathbb{Z}_{>0}$ such that the image of n in $\mathcal{O}(X)$ is invertible.
The following complex of sheaves is exact:

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{a \mapsto a^n} \mathbb{G}_m \to 0.$$

<u>Proof</u>: The nontrivial part is showing that $\mathbb{G}_m \xrightarrow{a \mapsto a^n} \mathbb{G}_m$ is surjective. This is equivalent to: Every local section $s \in \mathbb{G}_m(\mathcal{U})$ admits a cover $(\mathcal{U}_i \to X)_{i \in I}$ such that $s|_{\mathcal{U}_i}$ lifts under $a \mapsto a^n$, i.e. $s|_{\mathcal{U}_i}$ is an n'th power.
First cover $\mathcal{U}$ by affines $(\mathcal{U}_i)_i$, $\mathcal{U}_i = \mathrm{Spec}(R_i)$, $s|_{\mathcal{U}_i} =: s_i \ni R_i^\times$. Note that ~~since~~ we have $n \in R_i^\times$.
Therefore (Spec of) the map $R_i \to R_i[t]/(t^n - s_i)$ is étale.
So s lifts locally under $a \mapsto a^n$. □

<u>Rmk</u>: - This sequence is <u>not</u> exact in the Zariski topology.
- If $\zeta_n \in \mathcal{O}(X)$, then $\mu_n \cong \mathbb{Z}/n$. So this exact sequence might be useful for computing $H^i_{et}(X, \mathbb{Z}/n)$.

(Weil-divisor exact sequence)

<u>Lemma 4</u>: Let X be a regular, integral scheme.
Let $j: \eta \hookrightarrow X$ denote the inclusion of the gen. point. There is an exact sequence of sheaves on X_{et}:

$$0 \to \mathbb{G}_{m,X} \to j_* \mathbb{G}_{m,\eta} \to \bigoplus_{\substack{x \in X \\ \text{height } 1}} (i_x)_* \mathbb{Z} \to 0$$

<u>Proof</u>: This follows from the fact that if $\mathcal{U} \to X$ is étale, then $\mathcal{U}$ is regular and the exactness of the sequence in the Zariski topology (using that $\mathcal{O}_{X,x}$ is a UFD if X is regular) (□).

<u>Note</u>: If $X = C$ is a curve, i_x is a closed immersion, so $(i_x)_*$ is exact (see exercise sheet).

Lemma 5: For $f: Y \to X$ a morphism, f_* preserves injectives.

Proof: $\mathcal{I} \in Sh(Y_{et})$ is injective iff $Hom_{Sh(Y_{et})}(-, \mathcal{I})$ is exact.
By the adjunction (f^*, f_*) we have

$$Hom_{Sh(X_{et})}(-, f_*\mathcal{I}) = Hom_{Sh(Y_{et})}(f^*(-), \mathcal{I}),$$ which is exact since f^* is exact (see also exercise sheet) $\square$.

Corollary 1: For $C/\bar{k}$ a curve and $c_x : Spec(\bar{k}) \to C$ a point, we have
$H^i_{et}(C, (c_x)_* \mathcal{F}) = 0 \quad \forall \mathcal{F}, \ \forall i > 0.$

Pf: Pick an injective resolution $\mathcal{I}^\bullet$ for $\mathcal{F}$.
Then by Lemma 5, $(c_x)_*\mathcal{I}^\bullet$ is an injective resolution for $(c_x)_*\mathcal{F}$.
So we get $H^i_{et}(C, (c_x)_*\mathcal{F}) = h^i(\Gamma_C((c_x)_*\mathcal{I}^\bullet)) = h^i(\Gamma_{Spec(\bar{k})}(\mathcal{I}^\bullet)) = 0.$

for $i > 0$. $\qquad h^i(\Gamma_{Spec(\bar{k})}(\mathcal{I}^\bullet)) = H^i_{et}(Spec(\bar{k}), \mathcal{F})$ Alexander talk.

~~Now we wish to understand the term~~
~~Corollary 2:~~

Now we wish to understand the terms $H^i(C, j_*\mathbb{G}_{m,\eta})$.

Part 2: Leray spectral sequence.

Def: A (first-quadrant) spectral sequence in an abelian cat. $\mathcal{A}$ (starting at E_2) is the following data:

(a) A family of objects $(E_r^{p,q})_{p,q \in \mathbb{Z},\, r \geq 2}$, $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$.

(b) Morphisms: $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r, q-r+1}$ such that $d \circ d = 0$.
(e.g. for $r = 2$)

$$\begin{tikzcd}
 & 0 \arrow[rrd] & E_2^{0,2} \arrow[rrd] & & & \\
0 \arrow[rrd] & 0 & E_2^{0,1} \arrow[rrd] & E_2^{1,1} \arrow[rrd] & E_2^{2,1} & \\
 & 0 & E_2^{0,0} \arrow[rrd] & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} \\
 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

(c) $E_{r+1}^{p,q} = \ker(d_r^{p,q}) / \mathrm{Im}(d_r^{p-r, q+r-1})$
Note that for all (p,q), $\exists\, r_0 > 0$ s.t. $E_{r_0}^{p,q} = E_{r_0+m}^{p,q} \quad \forall m \geq 0$.
Denote $E_\infty^{p,q} := E_{r_0}^{p,q}$

(d) Objects $(E^n)_{n \geq 0}$ in $\mathcal{A}$ with filtrations $E^n = E^n_0 \supseteq E^n_1 \supseteq \dots \supseteq E^n_n \supseteq E^n_{n+1} = \{0\}$
and isomorphisms $E^n_p / E^n_{p+1} = E_\infty^{p, n-p}$. | Write $E_2^{p,q} \Rightarrow E^{p+q}$

(Grothendieck spectral sequence)

Theorem 1: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories and assume that $\mathcal{A}, \mathcal{B}$ have enough injectives. Let $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{C}$ be left exact. There is a spectral sequence $R^pG(R^qF(A)) \Rightarrow R^{p+q}(G\circ F)(A)$ for every object A of $\mathcal{A}$, under the condition: F sends injective objects to G-acyclic objects.

Proof: (See Weibel homological algebra, 5.8)

(Leray - spectral sequence)

Corollary 3: Let $f: X \to Y$ be a morphism of schemes. Let $\mathcal{F} \in Sh(X_{et})$. There is a spectral sequence

$$E_2^{p,q} = H^p_{et}(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}_{et}(X, \mathcal{F}).$$

Proof: Follows from Theorem 1 combined with Lemma 5, because injective objects are acyclic. $\square$

So for X integral with generic point $j: \eta \hookrightarrow X$, $H^p_{et}(X, j_* \mathbb{G}_{m,\eta}^{Sp(k)})$ is related to $H^p_{ct\text{-}gp}(\Gamma_k, (k^{sep})^\times)$.

Lemma 6: (Hilbert's Theorem 90)

For any field k we have $H^1_{ct\text{-}gp}(\Gamma_k, (k^{sep})^\times) = 0$.

Proof: Let L/k be finite Galois. We have

$H^1_{ct\text{-}gp}(\Gamma_k, (k^{sep})^\times) = \varinjlim_{\substack{M/k \\ fin.\ Gal.}} H^1_{gp}(Gal(M/k), M^\times)$, so it suffices

to show that $H^1_{gp}(Gal(L/k), L^\times) = 0$.

Let $\chi: Gal(L/k) \to L^\times$ be a 1-cocycle.

By independence of characters, the k-linear endomorphism $L \to L$ given by $\sum_{\sigma \in Gal(L/k)} \chi(\sigma) \cdot \sigma(-)$ is nonzero.

Let $\alpha \in L$ be such that $\underbrace{\sum_\sigma \chi(\sigma) \cdot \sigma(\alpha)}_{\text{"}\beta\text{"}} \neq 0$.

Then we have for any $\tau \in Gal(L/k)$, $\tau(\beta) =$

$$\sum_\sigma \tau(\chi(\sigma)) \cdot \tau\sigma(\alpha) = \sum_\sigma \chi(\tau)^{-1} \cdot \chi(\tau\sigma) \cdot \tau\sigma(\alpha) = \chi(\tau)^{-1} \cdot \beta.$$

So $\chi(\tau) = \frac{\beta}{\tau(\beta)} \Rightarrow \chi$ is a 1-coboundary $\square$

Chris' proof is better

Corollary 4: For an integral scheme X, we have $H^1_{et}(X, j_*\mathbb{G}_{m,\eta}) = 0$.
If X is regular we get $H^1(X, \mathbb{G}_m) = Cl(X)$.

Proof: We use the Leray spectral sequence $H^p_{et}(X, R^q j_* \mathbb{G}_{m,\eta}) \Rightarrow H^{p+q}_{Gal}(\Gamma_K, (K^{sep})^\times)$

We have $E_2^{1,0} = H^1_{et}(X, j_*\mathbb{G}_{m,\eta})$.

Outgoing maps: $E_r^{1,0} \xrightarrow{d_r^{1,0}} E_r^{r+1,-r+1} = 0$ are all zero $\forall r \geq 2$

Incoming maps $0 = E_r^{1-r,r-1} \xrightarrow{d_r^{1-r,r-1}} E_r^{1,0}$ are also zero. $\forall r \geq 2$.

So $E_2^{1,0} = E_\infty^{1,0}$.

So $E_\infty^{1,0}$ is a graded piece in a filtration of $H^1_{Gal}(\Gamma_K, (K^{sep})^\times)$

So $H^1_{et}(X, j_*\mathbb{G}_{m,\eta}) = 0$. $\qquad H^1_{Gal}(\Gamma_K, (K^{sep})^\times) = 0$ (H90).

For X regular, use Weil-divisor exact sequence:

$$H^1_{et}(X, \mathbb{G}_m) = \mathrm{Coker}\left(\bigoplus_{x \in X^{(1)}} \mathbb{Z} \xleftarrow{div} K^\times\right) = Cl(X). \quad \square$$

Corollary 5: Using the Kummer sequence for $\mathbb{A}^1$, $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ over $k = \bar{k}$,

$Pic(\mathbb{A}^1) = Cl(\mathbb{G}_m) = 0 \rightsquigarrow H^1(\mathbb{A}^1, \mu_n) = 0$, ~~$H^1(\mathbb{G}_m$~~,

$H^1(\mathbb{A}^1 \setminus \{0\}, \mu_n) = \mathrm{coker}\left(\bar{k}^\times \cdot \langle x \rangle_{\mathbb{Z}} \xrightarrow{a \mapsto a^n} \bar{k}^\times \cdot \langle x \rangle_{\mathbb{Z}}\right) = \mathbb{Z}/n$.

for rotation.

For C a curve over $\bar{k}$, we get more vanishing, $\eta = \mathrm{Spec}(K)$.

Theorem 2: Let K be the function field of a curve over $\bar{k}$.
Then we have $H^i_{Gal}(\Gamma_K, (K^{sep})^\times) = 0 \quad \forall i > 0$.

Proof: See (Serre Galois cohomology II. §3)

Ingredients:

- Def. A field L is C_i ($i \geq 0$) if for any $P \in L[X_1, \dots, X_N]$ homogeneous of degree d with $N > d^i$ has a nontrivial zero.

k is $C_0 \iff k = \bar{k}$.

- Tsen's theorem: k is C_i, M/k, tr. deg $(M/k) = n \Rightarrow M$ is C_{i+n}.
- $H^2(\Gamma_k, \bar{k}^\times) = \{\text{iso classes of CSA's}\} \ni [A]$.

A/k ($\dim d^2$) comes with reduced norm, a poly $N_{red} \in k[X_1, \dots, X_{d^2}]$ of degree d.

So k is $C_1 \Rightarrow N_{red}$ has nontrivial zero $\iff [A]$ is trivial.

- $H^2(\Gamma_k, (k^{sep})^\times) = 0 \Longrightarrow H^i(\Gamma_k, (k^{sep})^\times) = 0 \quad \forall i \geq 2$
(Serre Galois cohomology II.2.4).

For C a curve we get
$$H^p_{et}(C, R^q_{j_*} \mathbb{G}_{m,\eta}) \Rightarrow H^{p+q}_{Gal}(\Gamma_k, (k^{sep})^\times)$$

and $H^{p+q}_{Gal}(\Gamma_k, (k^{sep})^\times) = 0 \quad \forall\, p+q \geq 1$.
In particular, $E_\infty^{p,q}$ of spectral sequence vanishes if ~~...~~ $p \geq 1$ or $q \geq 1$.
For vanishing of $E_2^{p,0}$, $p \geq 1$, need $E_2^{p,0} = E_\infty^{p,0}$.
The outgoing differentials are zero.
So it suffices to show that $E_2^{p,q} = 0$ for all $q \geq 1$.

Lemma 7: Let $f : Y \to X$ be a morphism, $\mathcal{F} \in Sh(Y_{et})$.
The sheaf $R^q_{\pi_*}\mathcal{F}$ is the sheafification of the presheaf $U \mapsto H^q(U \times_X Y, \mathcal{F})$.
Proof: (on exercise sheet) (13).

So $R^q_{j_*}\mathbb{G}_{m,\eta}$ is the sheafification of the presheaf
$U \mapsto H^q_{et}(U \times_C \eta, \mathbb{G}_m)$.

$U \times_C \eta = \coprod_{j=1}^{n} \operatorname{Spec}(L_j)$, L_j / K finite, separable.

So $H^q_{et}(U \times_C \eta, \mathbb{G}_m) = \prod_{j=1}^{n} H^q_{Gal}(\Gamma_{L_j}, (L_j^{sep})^\times) = 0$, if $q \geq 1$.
Therefore we get ~~...~~
$H^p(C, j_*\mathbb{G}_m) = E_2^{p,0} = E_\infty^{p,0} = 0$.

We can now prove the key vanishing theorem.

Theorem 3: Let C be a smooth curve over $\bar{k}$. Then
$$H^i_{\acute{e}t}(C, \mathbb{G}_m) = 0 \quad \text{for all } i \geq 2.$$

Proof: Use the previously mentioned vanishing together with the Weil-divisor exact sequence □.

Corollary 6: Let C be a smooth connected curve of genus g over $\bar{k}$. We have:
$$H^i_{\acute{e}t}(C, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & \text{if } i = 0, 2 \\ (\mathbb{Z}/n)^{2g} & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}$$

Proof: (on exercise sheet). (□).