Weil Coh 5

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Čech cohomology

Definition: Let $\mathcal{P}$ be a presheaf on a scheme X.
Let $\mathcal{U} = (U_i)_i$ be an étale cover of X. For every $(p+1)$-tuple $(i_0, \dots, i_p)$ we write $U_{i_0} \times_X \dots \times_X U_{i_p} = U_{i_0 \dots i_p}$.
Define the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{P})$ as follows:

$$\check{C}^p(\mathcal{U}, \mathcal{P}) = \prod_{i_0, \dots, i_p} \mathcal{P}(U_{i_0 \dots i_p})$$

$$d^p : \check{C}^p(\mathcal{U}, \mathcal{P}) \longrightarrow \check{C}^{p+1}(\mathcal{U}, \mathcal{P})$$
$$(s_{i_0 \dots i_p})_{i_0, \dots, i_p} \longmapsto \Big(\sum_{j=0}^{p+1} (-1)^j s_{i_0 \dots i_{j-1} i_{j+1} \dots i_{p+1}}|_{U_{i_0 \dots i_{p+1}}} \Big)_{i_0, \dots, i_{p+1}}$$

Its cohomology groups $\check{H}^p(\mathcal{U}/X, \mathcal{P})$ are called the Čech cohomology groups.

Example: $\check{H}^0(\mathcal{U}/X, \mathcal{P}) = \ker \Big(d^0 : \prod_i \mathcal{P}(U_i) \to \prod_{i,j} \mathcal{P}(U_{ij}),\ (s_i)_i \mapsto (s_i|_{U_{ij}} - s_j|_{U_{ij}})_{i,j} \Big)$
so $\check{H}^0(\mathcal{U}/X, \mathcal{P}) = \mathcal{P}(X)$ if $\mathcal{P}$ is a sheaf.

Example: $\check{H}^1(\mathcal{U}/X, \mathcal{P})$ consists of $(s_{ij})_{i,j}$ satisfying the 1-cocycle condition $s_{ij} + s_{jk} = s_{ik}$ on U_{ijk} for all i,j,k, modulo the 1-coboundaries $(\sigma_i|_{U_{ij}} - \sigma_j|_{U_{ij}})_{i,j}$.

Definition: With notation as above, a second étale cover $\mathcal{V} = (V_j)_j$ is called a refinement of $\mathcal{U}$ if for each j, the map $V_j \to X$ factors through $U_{i_j} \to X$ for some i_j.

Note: This gives well defined maps

$$\rho(\mathcal{U}, \mathcal{V}) : \check{H}^p(\mathcal{U}/X, \mathcal{P}) \longrightarrow \check{H}^p(\mathcal{V}/X, \mathcal{P})$$
$$[(s_{i_0 \dots i_p})_{i_0, \dots, i_p}] \mapsto [(s_{i_{j_0} \dots i_{j_p}}|_{U_{i_{j_0} \dots i_{j_p}}})_{j_0, \dots, j_p}]$$

that are independent of the choice of factorizations.

Definition: The Čech cohomology groups of a presheaf $\mathcal{P}$ on X are

$$\check{H}^p(X, \mathcal{P}) = \varinjlim \check{H}^p(\mathcal{U}, \mathcal{P})$$

where the limit is over all coverings $\mathcal{U}$ of X, cofiltered by refinement.

Proposition: For every p, we have

$$R^p \check{H}^0(\mathcal{U}/X, \mathcal{P}) = \check{H}^p(\mathcal{U}/X, \mathcal{P})$$
$$R^p \check{H}^0(X, \mathcal{P}) = \check{H}^p(X, \mathcal{P})$$

Proof: See Milne III.2.7.

Exercise: Let $X = \operatorname{Spec} k$. For ℓ/k Galois, we have an étale cover $(U \to X)$ with $U = \operatorname{Spec} \ell$. Show that $\check{H}^p(U/X, \mathcal{F}) = H^p(\mathrm{Gal}(\ell/k), \mathcal{F}(U))$.
Conclude that $\check{H}^p(X, \mathcal{F}) = H^p(\Gamma_k, \mathcal{F}(\operatorname{Spec} k^{sep}))$.
Note that this is exactly $H^p(X, \mathcal{F})$.

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Comparison with derived functor cohomology

Lemma: The inclusion functor $i : Sh(X_{ét}) \longrightarrow PreSh(X_{ét})$ is left-exact, preserves injectives and has right derived functors $\mathcal{H}^p$ given by

$$\mathcal{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

Proof: Sheafification $(-)^\#$ is exact and left-adjoint to i, so i is left-exact and preserves injectives (since $\mathrm{Hom}(-, i\mathcal{I}) = \mathrm{Hom}((-)^\#, \mathcal{I})$ is exact for injective $\mathcal{I}$). For an injective resolution $\mathcal{F} \longrightarrow \mathcal{I}^\bullet$, both $\mathcal{H}^p(\mathcal{F})(U)$ and $H^p(U, \mathcal{F})$ are the cohomology of $\mathcal{I}^\bullet(U)$. $\square$

Proposition: For an étale cover $\mathcal{U}$ of X and $\mathcal{F} \in Sh(X_{ét})$ we have spectral sequences

$$\check{H}^p(\mathcal{U}/X, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$
$$\check{H}^p(X, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Proof: This follows from the Grothendieck spectral sequence, as $\check{H}^0(\mathcal{U}/X, -) \circ i = \Gamma(X, -)$ and $\check{H}^0(X, -) \circ i = \Gamma(X, -)$ and i sends injectives to injectives, which are acyclic. $\square$

Note: Cohomology is locally trivial, i.e. for every $\alpha \in H^q(X, \mathcal{F})$ there exists an étale cover $(U_i)_i$ of X such that α maps to 0 in each $H^p(U_i, \mathcal{F})$. To see this, note that this is equivalent to $\mathcal{H}^q(\mathcal{F})^\# = 0$. Indeed, if $\mathcal{F} \longrightarrow \mathcal{I}^\bullet$ is an injective resolution, then $\mathcal{H}^q(\mathcal{F})^\#$ is the cohomology of $i\mathcal{I}^{\bullet\#} = \mathcal{I}^\bullet$, which is trivial. Hence, we have $\check{H}^0(X, \mathcal{H}^q(\mathcal{F})) = 0$ and the spectral sequence gives

$$\check{H}^0(X, \mathcal{F}) = H^0(X, \mathcal{F})$$
$$\check{H}^1(X, \mathcal{F}) = H^1(X, \mathcal{F})$$

and an exact sequence

$$0 \longrightarrow \check{H}^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow \check{H}^1(X, \mathcal{H}^1(\mathcal{F})) \longrightarrow \check{H}^3(X, \mathcal{F}) \longrightarrow H^3(X, \mathcal{F}).$$

Torsors

Definition: Let $\mathcal{F} \in Sh(X_{ét})$ and suppose $\mathcal{T}$ is a sheaf of sets on $X_{ét}$ on which $\mathcal{F}$ acts from the right. We call $\mathcal{T}$ an <u>$\mathcal{F}$-torsor</u> if the following hold:
- there exists an étale cover $(U_i)_i$ of X such that $\mathcal{T}(U_i) \neq \emptyset$ for every i
- whenever $U \longrightarrow X$ is étale with $s \in \mathcal{T}(U)$, the map $\mathcal{F}|_U \longrightarrow \mathcal{T}|_U$, $g \longmapsto sg$ is an isomorphism of sheaves of sets.

- whenever $U \to X$ is étale with $s \in T(U)$, the map $\mathcal{F}|_U \to T|_U$, $g \mapsto sg$ is an isomorphism of sheaves of sets.

An isomorphism of $\mathcal{F}$-torsors is an isomorphism of the underlying sheaves of sets commuting with the $\mathcal{F}$-action.

Exercise: Let $\mathcal{F} \in Sh(X_{ét})$. Show that there is a bijection

$$\{\mathcal{F}\text{-torsors}\}/\cong \xrightarrow{\sim} \check{H}^1(X, \mathcal{F})$$

given by sending T to $(g_{ij})_{i,j} \in \check{C}(\mathcal{U}, \mathcal{F})$, where $\mathcal{U} = (U_i)_i$ is an étale cover of X with sections $s_i \in T(U_i)$ and $g_{ij} \in \mathcal{F}(U_{ij})$ is the unique element such that $s_i|_{U_{ij}} \cdot g_{ij} = s_j|_{U_{ij}}$.

Remark: If $\mathcal{F}$ is represented by an affine commutative group scheme, then every $\mathcal{F}$-torsor is represented by a scheme.

Exercise: (1) Let X be connected and G a finite group. Show that G_X-torsors are represented by Galois $Y \to X$ with Galois group G. Conclude: $H^1(X_{ét}, G_X) \cong Hom_{cont}(\pi_1(X, \bar{x}), G)$.

(2) For R a PID and an integer $n \in R^\times$, show that μ_n-torsors are exactly $Spec\, R[T]/(T^n - u)$ with $u \in R^\times/(R^\times)^n$.

(3) Deduce Kummer Theory: for a field k with $n \in k^\times$, $\zeta_n \in k$, there is a bijection

$$\begin{aligned} k^\times/(k^\times)^n &\xrightarrow{\sim} \{\text{cyclic degree } n \text{ extensions of } k\} \\ u &\longmapsto k(\sqrt[n]{u})/k \end{aligned}$$

Comparison Theorem

Theorem: Let X be a quasi-compact scheme such that every finite subset of X is contained in an open affine set (for example, X quasi-projective over an affine scheme) and $\mathcal{F} \in Sh(X_{ét})$. Then there are canonical isomorphisms

$$\check{H}^p(X_{ét}, \mathcal{F}) \xrightarrow{\sim} H^p(X_{ét}, \mathcal{F})$$

for all p.

Cup product

Proposition: Let $\mathcal{F}, \mathcal{G} \in Sh(X_{ét})$. There is a pairing

$$\cup : \check{H}^r(X, \mathcal{F}) \times \check{H}^s(X, \mathcal{G}) \to \check{H}^{r+s}(X, \mathcal{F} \otimes \mathcal{G}).$$
$$([(f_{i_0 \cdots i_r})_{i_0, \ldots, i_r}], [(g_{i_0 \cdots i_s})_{i_0, \ldots, i_s}]) \mapsto [(f_{i_0 \cdots i_r} \otimes g_{i_{r+1} \cdots i_{r+s}})_{i_0, \ldots, i_{r+s}}]$$

natural in $\mathcal{F}$ and $\mathcal{G}$. For a morphism $f : Y \to X$ and $\alpha \in H^r(X, \mathcal{F})$ and $\beta \in H^s(X, \mathcal{G})$ we have $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$.

Definition: We call this the <u>cup-product</u>.

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Remark: If $X = \operatorname{Spec} k$, then the cup-product coincides with the cup-product in Galois cohomology. If X is a smooth proper curve over an algebraically closed field of characteristic not dividing n, then the cup-product agrees with the e_n-pairing

$$\mathrm{Jac}(X)[n] \times \mathrm{Jac}(X)[n] \longrightarrow \mu_n(k).$$

Theorem: (Poincaré duality)
Let X be a proper smooth variety of dimension d over an algebraically closed field k of characteristic coprime to n. Let $\mathcal{F}$ be a sheaf on $X_{\acute{e}t}$ that is locally a constant sheaf defined by a finite $\mathbb{Z}/n\mathbb{Z}$-module. Then we have the following.

- There is a canonical isomorphism

$$H^{2d}(X, \mu_n^{\otimes d}) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$$

- The cup product gives a perfect pairing of finite groups

$$H^r(X, \mathcal{F}) \times H^{2d-r}(X, \mathcal{F}^\vee(d)) \longrightarrow H^{2d}(X, \mu_n^{\otimes d}) \cong \mathbb{Z}/n\mathbb{Z}$$

where $\mathcal{F}^\vee(d) = \mathcal{H}om(\mathcal{F}, \mu_n^{\otimes d})$.