INTRODUCTION: ZETA FUNCTION AND WEIL COHOMOLOGIES

OLD AND NEW PROBLEMS

- LET $p \equiv 1 \bmod 3$, DETERMINE NUMBER OF SOLUTIONS ON $\mathbb{F}_p$ OF

$$ax^3 + by^3 = 1$$

(DISQUISITIONES ARITHMETICAE - GAUSS 1798)

- HOW MANY NONZERO POINTS ARE THERE ON

$$x^3y + y^3z + z^3x = 0$$

OVER $\mathbb{F}_{5^{18}}$ UP TO SCALING?

(FRONTIER MATH - EPOCH AI 2025)

ANDRÉ WEIL (1906 - 1998) FRENCH MATEMATICIAN - FOUNDATIONAL WORK IN NT AND AG

- STUDIED AT ÉCOLE NORMALE SUPÉRIEUR
  W/ HADAMARD AND PICARD
- BROTHER OF SIMONE WEIL FAMOUS FRENCH PHILOSOPHER
- 1940 REFUSED TO SERVE IN THE ARMY
  - 3 MONTHS OF PRISON IN ROUEN (FEBRUARY - MAY)

FRAGMENT OF LETTER TO SIMONE, MARCH 26, 1940

"THE THOUGHTS THAT FOLLOW ARE OF TWO SORTS.
THE FIRST CONCERN THE HISTORY OF THE THEORY OF NUMBERS; YOU MAY BE ABLE TO UNDERSTAND THE BEGINNING; YOU WILL UNDERSTAND NOTHING OF WHAT IT FOLLOWS THAT.
THE OTHER CONCERN THE ROLE OF ANALOGY IN MATHEMATICAL DISCOVERY, EXAMINING A PARTICULAR EXAMPLE, AND PERHAPS YOU WILL BE ABLE TO PROFIT FROM IT."

"[...] MY WORK CONSISTS IN DECIPHERING A TRILINGUAL TEXT (ROSETTA STONE) OF EACH OF THE THREE COLUMNS I HAVE ONLY DISPARATE FRAGMENTS, I HAVE SOME IDEAS ABOUT EACH OF THE THREE LANGUAGES: BUT I KNOW AS WELL THERE ARE GREAT DIFFERENCES IN MEANING FROM ONE COLUMN TO ANOTHER FOR WHICH NOTHING HAS PREPARED ME IN ADVANCE"

TRILINGUAL TEXT

- NUMBER FIELDS
- FUNCTION FIELDS
- 'RIEMANNIAN FIELDS' ~ VARIETIES OVER $\mathbb{C}$.

CONCLUDING THE LETTER 14 PAGES AFTER WITH A POST SCRIPTUM

"I SEND THIS TO YOU WITHOUT REREADING, I FEAR HAVING MADE MORE OF MY RESEARCH THAN I INTENDED, THAT IS, IN ORDER TO EXPLAIN (FOLLOWING YOUR REQUEST) HOW ONE DEVELOPS ONE'S RESEARCH, I HAVE BEEN FOCUSING ON THE LOCKS I WISH TO OPEN".

§ 'NUMBERS OF SOLUTIONS OF EQUATIONS IN FINITE FIELD' - 1948

CLASSICAL PROBLEM OF COUNTING RATIONAL SOLUTIONS

LET $k = \mathbb{F}_q$ FINITE FIELD W/ q ELEMENTS $q = p^r$ p PRIME

$k_s = \mathbb{F}_{q^s}$ UNIQUE EXTENSION OF k W/ q^s ELEMENTS

<u>EG</u>

- $X = \mathbb{P}^1_k$ $\quad \#X(k) = q+1$

  $\#X(k_s) = q^s + 1$

- $X = \mathbb{P}^n_k$ $\quad \mathbb{P}^n_k = \mathbb{A}^n_k \sqcup \mathbb{P}^{n-1}_k$

  $\#X(k) = q^n + q^{n-1} + \cdots + q + 1$

  $\#X(k_s) = q^{sn} + q^{s(n-1)} + \cdots + q^s + 1$

- $X = \operatorname{Spec}\left(k[x,y]/(x^2+y^2-1)\right)$ CIRCLE

  $p \neq 2$, $\chi : \mathbb{F}_q^\times \longrightarrow \{\pm 1\}$ QUADRATIC CHARACTER $\chi(a) = \begin{cases} 1 & a \text{ IS SQUARE} \\ -1 & \text{OTHERWISE} \end{cases}$

  $\#X(k) = q - \chi(-1)$

  $\#X(k_s) = q^s - \chi_s(-1)$

  $$q - \chi(-1) = \begin{cases} q-1 & \text{IF } q \equiv 1 \mod 4 \\ q+1 & \text{IF } q \equiv 3 \mod 4 \end{cases}$$

  $p = 2$ $\quad x^2 + y^2 - 1 = 0 \Rightarrow x + y = 1$

  $\#X(k) = q$

  $\#X(k_s) = q^s$

- $X = C$ ELLIPTIC CURVE OVER k

  $\varphi : X \longrightarrow X$ FROBENIUS ENDOMORPHISM

  $(x,y) \longmapsto (x^q, y^q)$

  $\#X(k) = \# \ker(\mathrm{Id} - \varphi) = \deg(\mathrm{Id} - \varphi)$ SINCE $\mathrm{Id} - \varphi$ SEPARABLE

$$[\deg(1-\varphi)] = (1-\varphi)\circ\widehat{(1-\varphi)} = (1-\varphi)\circ(1-\hat{\varphi}) =$$
$$= 1-\varphi-\hat{\varphi}+\varphi\circ\hat{\varphi} = 1+q-\varphi-\hat{\varphi}$$

$[t] = \varphi+\hat{\varphi}$, t TRACE OF FROBENIUS

$X^2 - tX + q$ CHARACTERISTIC POLYNOMIAL OF FROBENIUS

α, α' ROOTS OF POLYNOMIAL $t = \alpha + \alpha'$

$$\# X(k) = 1+q-t = 1+q-\alpha-\alpha'$$

ANALOGOUSLY

$$\# X(k_s) = \# \ker(\mathrm{Id}-\varphi^s) = \deg(1-\varphi^s)$$

$$[\deg(1-\varphi^s)] = 1+q^s-\varphi^s-\hat{\varphi}^s$$

$$\# X(k_s) = 1+q^s-\alpha^s-\alpha'^s$$

INSPIRED BY GAUSS'S WORK IN 'DISQUISITIONES ARITHMETICAE', WEIL STUDIED FOLLOWING FAMILIES OF HYPERSURFACES

(*) $a_0 x_0^{n_0} + a_1 x_1^{n_1} + \dots + a_r x_r^{n_r} = 0$ $\quad n_i > 0,\ a_i \in \mathbb{F}_q$

FERMAT HYPERSURFACE

SKETCH OF WEIL'S APPROACH:

- MAIN IDEA: ORTHOGONALITY RELATIONS OF CHARACTERS
- N NUMBER OF SOLUTIONS IN k OF (*)

  LET $u \in k$, $N_i(u)$ NUMBER OF SOLUTIONS OF $x^{n_i} = u$
- LET $L((u_i)_i) = \sum_{i=0}^{r} a_i u_i$

$$N = \sum_{L(u)=0} N_0(u_0)\cdots N_r(u_r)$$

- ORTHOGONALITY RELATION $N_i(u) = \sum_{\alpha} \chi_\alpha(u)$

  χ_α MULTIPLICATIVE CHARACTER

  $\chi_\alpha : k^* \longrightarrow \mathbb{C}^*$, $\quad n_i\alpha \equiv 0 \mod 1 \quad 0 \le \alpha < 1$

- $N = q^r + \sum_{u,\alpha} \chi_{\alpha_0}(u_0)\cdots\chi_{\alpha_r}(u_r)$ SUMMING OVER
  - $L(u) = 0$
  - $d_i\alpha_i \equiv 0 \mod 1 \quad 0 \le \alpha < 1$

THEOREM (WEIL)

CONSIDER THE FERMAT HYPERSURFACE

$$a_0 x_0^{n_0} + a_1 x_1^{n_1} + \cdots + a_r x_r^{n_r} = 0 \qquad n_i > 0, \quad a_i \in \mathbb{F}_q$$

THEN THE NUMBER OF POINTS OVER $\mathbb{F}_q$ IS GIVEN BY

$$q^r + \frac{q-1}{q} \sum_{(\chi_0, \ldots, \chi_r)} \chi_0(a_0)^{-1} \cdots \chi_r(a_r)^{-1} g(\chi_0) \cdots g(\chi_r)$$

WHERE $(\chi_0, \ldots, \chi_r)$ RUNS OVER ALL TUPLES IN WHICH χ_i IS MULTIPLICATIVE CHARACTER OF $\mathbb{F}_q$ OF ORDER DIVIDING $\gcd(n_i, q-1)$ AND $\chi_0 \cdots \chi_r = 1$.

REM FOR χ NON-TRIVIAL MULTIPLICATIVE CHARACTER OF χ AND $\psi : k \xrightarrow[\text{trace}]{} \mathbb{F}_p \longrightarrow \mathbb{C}^\times$, $x \longmapsto \exp(2\pi i x/p)$ ADDITIVE CHARACTER

THE GAUSS SUM ASSOCIATED TO χ IS GIVEN BY

$$g(\chi) := \sum_{x \in k} \chi(x) \psi(x) \in \mathbb{C}.$$

WEIL USE FUNDAMENTAL PROPERTY OF GAUSS SUM TO EXTEND COMPUTATION TO k_s-POINTS

DAVENPORT - HASSE

FOR AN EXTENSION $\mathbb{F}_{q^s}$ OF $\mathbb{F}_q$ PUT $\chi' := \chi \circ \mathrm{Norm}_{\mathbb{F}_{q^s}/\mathbb{F}_q}$

$\psi' := \psi \circ \mathrm{Trace}_{\mathbb{F}_{q^s}/\mathbb{F}_q}$

THEN

$$-g(\chi') = (-g(\chi))^s. \quad \rightsquigarrow \text{ CAN WRITE } \quad \# X(k_s) = \sum \pm \alpha_i^s$$

WEIL'S IDEA: CONSIDER GENERATING POWER SERIES FOR NUMBER OF RATIONAL POINTS

$$\sum_{s \geq 1} (\# X(k_s)) \cdot U^{s-1}$$

EG

- $X = \mathbb{P}^1_k$, $\quad \displaystyle\sum_{s \geq 1} (q^s + 1) U^{s-1} = \frac{d}{dU} \log\left(\frac{1}{1-qU}\right) + \frac{d}{dU} \log\left(\frac{1}{1-U}\right)$

$$= \frac{d}{dU} \log\left(\frac{1}{(1-U)(1-qU)}\right)$$

- $X = \mathbb{P}^n_k$ $\quad \displaystyle\sum_{s \geq 1} (q^{sn} + q^{s(n-1)} + \cdots + q^s + 1) = \frac{d}{dU} \log\left(\frac{1}{(1-U)(1-qU)\cdots(1-q^nU)}\right)$

- $X = C$ $\quad \displaystyle\sum_{s \geq 1} (1 + q^s - \alpha^s - \alpha'^s) = \frac{d}{dU} \log\left(\frac{(1-\alpha U)(1-\alpha' U)}{(1-U)(1-qU)}\right)$

$$= \frac{d}{dU} \log\left(\frac{1 - tU + qU^2}{(1-U)(1-qU)}\right)$$

WEIL FOUND THAT FOR THE FERMAT'S HYPERSURFACE

$$\sum_{s\geq 1} (\#\tilde{X}(k_s))U^{s-1} = \sum_{n=0}^{r-1} \frac{d}{dU} \log\left(\frac{1}{(1-q^n U)}\right) +$$

$$+ (-1)^r \sum_{\alpha} \frac{1}{\mu(\alpha)} \frac{d}{dU} \log\left(1 - C(\alpha)\, U^{\mu(\alpha)}\right)$$

SUMMING OVER $n\alpha_i \equiv 0$, $\sum \alpha_i \equiv 0 \mod 1$, $0 < \alpha_i < 1$

$$C(\alpha) = (-1)^{r-1} \chi_{\alpha_0}(\bar{a}_0^{-1}) \cdots \chi_{\alpha_r}(\bar{a}_r^{-1})\, g(\chi_{\alpha_0}) \cdots g(\chi_{\alpha_r})$$

"THIS AND OTHER EXAMPLES WHICH WE CANNOT DISCUSS HERE, SEEM TO LEND SOME SUPPORT TO THE FOLLOWING CONJECTURAL STATEMENTS, WHICH ARE KNOWN TO BE TRUE FOR CURVES, BUT WHICH I HAVE NOT SO FAR BEEN ABLE TO PROVE FOR VARIETIES OF HIGHER DIMENSION"

WEIL 1948

§ FORMULATION OF THE CONJECTURES

LET $k = \mathbb{F}_q$ FINITE FIELD OF q - ELEMENTS

X ~~QUASI~~ PROJECTIVE ALGEBRAIC VARIETY NON-SINGULAR

DEFINE THE <u>ZETA FUNCTION</u> OF X

$$Z(X,T) := \exp\left(\sum_{r=1}^{\infty} \frac{T^r}{r} \#X(k_r)\right) \in \mathbb{Z}[[T]]$$

<u>THEOREM</u> (WEIL CONJECTURES)

THE SERIES $Z(X,T)$ HAS THE FOLLOWING PROPERTIES :

1. RATIONALITY

   THE SERIES $Z(X,T)$ REPRESENTS A RATIONAL FUNCTION OF T

2. FUNCTIONAL EQUATION

   SUPPOSE X IS OF PURE DIMENSION n

   $$Z\left(X, \frac{1}{q^n T}\right) = \pm q^{nE/2}\, T^E\, Z(X,T) \quad \text{FOR SOME INTEGER } E$$

3. RIEMANN HYPOTHESIS

   THERE IS A UNIQUE FACTORIZATION

   $$Z(X,T) = \frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)}$$

   IN WHICH $P_i(T)$ FACTORS OVER $\mathbb{C}$ AS $\prod_j (1 - \alpha_{ij} T)$ WHERE $|\alpha_{ij}| = q^{i/2}$ FOR EACH j.

IN PARTICULAR, THE INTEGER E EQUALS

$$E = \sum (-1)^i \deg P_i$$

IF X IS GEOMETRICALLY IRREDUCIBLE THEN $P_0(T) = 1 - T$,
$P_{2n}(T) = 1 - q^n T$.

4. BETTI NUMBERS

SUPPOSE IN ADDITION THERE EXIST A NUMBER FIELD K,
A FINITE SET OF PRIME IDEALS S OF $\mathcal{O}_K$,
A MAXIMAL IDEAL $\mathfrak{p} \subset \mathcal{O}_K$ NOT CONTAINED IN S, $\mathcal{O}_K/\mathfrak{p} \simeq k$
A SMOOTH PROJECTIVE SCHEME $\mathfrak{X}$ OVER $\mathcal{O}_{K,S}$
SUCH THAT $X \simeq \mathfrak{X} \times_{\mathcal{O}_{K,S}} k$ BASE CHANGE, I.E X IS REDUCTION mod $\mathfrak{p}$
THEN FOR ANY EMBEDDING $K \to \mathbb{C}$, THE i-TH BETTI NUMBER OF THE TOPOLOGICAL SPACE $(\mathfrak{X} \times_{\mathcal{O}_{K,S}} \mathbb{C})^{an}$ EQUALS $\deg P_i$.

REM WEIL CONJECTURES FOR $\mathbb{P}^1$ AND C

$$Z(\mathbb{P}^1, T) = \frac{1}{(1-T)(1-qT)}, \qquad Z(C,T) = \frac{1 - tT + qT^2}{(1-T)(1-qT)}$$

- RATIONALITY, OBVIOUS
- FUNCTIONAL EQUATION

$$Z\left(\mathbb{P}^1, \frac{1}{qT}\right) = \frac{qT^2}{(1-T)(1-qT)} \qquad E = 2$$

$$Z\left(C, \frac{1}{qT}\right) = \frac{1 - tT + qT^2}{(1-T)(1-qT)} \qquad E = 0$$

- RIEMANN HYPOTHESIS

FACTORIZATION FOR $\mathbb{P}^1$ OBVIOUS

FACTORIZATION FOR C

$$P_i(T) = \begin{cases} 1 - T & i = 0 \\ 1 - tT + qT^2 & i = 1 \\ 1 - qT & i = 2 \end{cases}$$

RIEMANN HYPOTHESIS ROOTS OF P_1 LIE ON CIRCLE $|\alpha| = q^{1/2}$

$\rightsquigarrow$ EQUIVALENT TO HASSE BOUND

- BETTI NUMBERS

$\mathbb{P}^1$: 1, 0, 1 $\qquad$ C: 1, 2, 1.

§ PREHISTORY OF WEIL CONJECTURES

- NUMBER FIELD SETTING MID-LATE 1800 ~ 1870s

K NUMBER FIELD, $\mathcal{O}_K$ RING OF INTEGERS (INTEGRAL CLOSURE OF $\mathbb{Z}$ IN K)

THE DEDEKIND ZETA FUNCTION OF K IS THE FORMAL EXPRESSION

$$\zeta_K(s) := \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{p})^{-s}}$$

PRODUCT OVER ALL MAXIMAL IDEALS

$$\mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$$

FOR $s \in \mathbb{C}$, $\mathrm{Re}\, s > 1$ THE PRODUCT CONVERGES ABSOLUTELY

$$\zeta_K(s) = \sum_{I \subseteq \mathcal{O}_K} \mathrm{Nm}_{K/\mathbb{Q}}(I)^{-s}$$

HECKE: FUNCTION $\zeta_K(s)$ EXTENDS MEROMORPHICALLY TO $\mathbb{C}$, W/ A SIMPLE POLE AT $s=1$ AND NO OTHER POLES.

RIEMANN HYPOTHESIS: ALL NONTRIVIAL ZEROES OF $\zeta_K(s)$ LIE ON THE LINE $\mathrm{Re}\, s = \frac{1}{2}$

- FUNCTION FIELD SETTING

FIX FINITE FIELD $\mathbb{F}_q$, LET K FUNCTION FIELD OVER $\mathbb{F}_q$, I.E. FINITE-DEGREE EXTENSION OF FIELD OF RATIONAL FUNCTIONS $\mathbb{F}_q(t)$.

LET $\mathcal{O}_K$ INTEGRAL CLOSURE OF $\mathbb{F}_q[t]$ IN K

$$\zeta_K(s) := \prod_{\mathfrak{p} \subset \mathcal{O}_K} \frac{1}{1 - \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{p})^{-s}}$$

IN GEOMETRIC LANGUAGE, X NORMALIZATION OF $\mathbb{A}^1_{\mathbb{F}_q}$ IN K

$\rightsquigarrow$ X UNIQUE NONSINGULAR PROJECTIVE CURVE / $\mathbb{F}_q$ W/ FUNCTION FIELD K

X° SET OF CLOSED POINTS IN K

$$\zeta_K(s) = \prod_{P \in X^\circ} \frac{1}{1 - \#\kappa(P)^{-s}} = \prod_{P \in X^\circ} \frac{1}{1 - q^{-d_P s}} \qquad d_P = [\kappa(P) : \mathbb{F}_q]$$

$$\zeta_K(s) = \exp\left(\sum_{n=1}^{\infty} \frac{q^{-ns}}{n} \# X(\mathbb{F}_{q^n}) \right)$$

CALLED HASSE-WEIL ZETA FUNCTION

REM BETWEEN 1940 AND 1946, WEIL PROVED AND FORMALIZED THE RIEMANN HYPOTHESIS IN THE FUNCTION FIELD CASE, I.E. FOR CURVES. ALONG THE WAY, HE NEEDED TO FORMALIZE, SOMETIMES FROM SCRATCH THE THEORY OF ALGEBRAIC VARIETIES OVER A NONZERO CHARACTERISTIC FIELD

$\sim$ 'FOUNDATIONS OF ALGEBRAIC GEOMETRY' - WEIL 1946

§ FROM CONJECTURES TO THEOREMS

- RATIONALITY PROVED BY DWORK 1958

  P-ADIC ANALYTIC METHODS

- MODERN FOUNDATIONS OF ALGEBRAIC GEOMETRY, GROTHENDIECK 1960'S

  THEORY OF ÉTALE COHOMOLOGY

  - NEW PROOF OF RATIONALITY - VIA LEFSCHETZ TRACE FORMULA
  - FIRST PROOF OF FUNCTIONAL EQUATION - VIA POINCARÉ DUALITY
  - FIRST PROOF OF BETTI NUMBERS - VIA COMPARISON THEOREMS

- RIEMANN HYPOTHESIS PROVED BY DELIGNE 1970s

  MORE AD HOC APPROACH

- GROTHENDIECK PROPOSED APPROACH OF RIEMANN HYPOTHESIS VIA "STANDARD CONJECTURES", STILL OPEN.

- DWORK METHODS ADAPTED TO P-ADIC ANALYTIC COHOMOLOGY, 1970S

  PROOF VIA ÉTALE COHOMOLOGY CAN BE EMULATED

  - RIGID COHOMOLOGY
  - CRYSTALLINE COHOMOLOGY

§ A COHOMOLOGICAL APPROACH

IDEA/ INSPIRATION: COUNTING FIXED POINTS OF A SELF MAP

$\updownarrow$

COMPUTING TRACES OF ASSOCIATED LINEAR MAP

- σ PERMUTATION OF $\{1,..,n\}$ ~ NUMBER OF FIXED POINT = EQUAL TO TRACE OF PERMUTATION MATRIX

- $f: X \longrightarrow X$ CONTINUOUS MAP, X COMPACT TRIANGLABLE SPACE

$$\sum(-1)^i \, \mathrm{Trace}\,(f, H^i(S))$$ LEFSCHETZ TRACE FORMULA

GIVES WEIGHTED COUNT OF FIXED POINTS OF f.

WEIL PROPOSED THE EXISTENCE OF COHOMOLOGY THEORY FOR WHICH AN ANALOGOUS OF LEFSCHETZ TRACE FORMULA WOULD HOLD.

METACONJECTURE (WEIL COHOMOLOGIES)

FOR SOME FIELD K, THERE EXISTS A SERIES OF CONTRAVARIANT COHOMOLOGY FUNCTORS

$$H^i : \{\text{ALGEBRAIC VARIETIES } / \mathbb{F}_q\} \longrightarrow \{\text{FINITE DIM. VECTOR SPACES } / K\}$$

SATISFYING THE FOLLOWING PROPERTIES:

i) CUP-PRODUCT STRUCTURE

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X)$$

ii) POINCARÉ DUALITY

$$H^i(X) \times H^{2d-i}(X) \longrightarrow K(-d)$$

W/ $d = \dim X$ AND $K(-d)$ TWISTED K BY F-ACTION SENDING 1 TO q^d

iii) LEFSCHETZ FIXED POINT FORMULA

$$\# X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d} (-1)^i \operatorname{Trace}(F^n \mid H^i(X))$$

W/ ABUSE OF NOTATION, $F^n : X \longrightarrow X$ n-TH POWER OF FROBENIUS

iv) COMPARISON THEOREM

$$H^i(X) \otimes_K \mathbb{C} \simeq H^i_{sing}((X \times_K \mathbb{C})^{an}, \mathbb{C})$$

REM ÉTALE AND CRYSTALLINE ARE TWO EXAMPLES OF WEIL COHOMOLOGIES.

§ APPENDIX - LINEAR ALGEBRAIC LEMMAS

LEMMA LET φ ENDOMORPHISM OF FINITE DIMENSIONAL VECTOR SPACE V OVER FIELD K. THEN WE HAVE IDENTITY OF FORMAL POWER SERIES IN T W/ COEFF. IN K

$$\exp\left(\sum_{r=1}^{\infty} \operatorname{Trace}(\varphi^r \mid V) \frac{T^r}{r}\right) = \det(1 - \varphi T \mid V)^{-1}.$$

LEMMA LET $V \times W \longrightarrow K$ PERFECT PAIRING OF VECTOR SPACES V, W OF dim r OVER K. LET $\lambda \in K$ AND LET $\varphi : V \longrightarrow V$ AND $\psi : W \longrightarrow W$ BE ENDOMORPHISMS SUCH THAT

$$\langle \varphi v, \psi w \rangle = \lambda \langle v, w \rangle \quad \forall\, v \in V,\ w \in W$$

THEN

$$\det(1 - \psi T \mid W) = \frac{(-1)^r \lambda^r T^r}{\det(\varphi \mid V)} \det\left(1 - \frac{\varphi}{\lambda T} \mid V\right)$$

AND

$$\det(\psi \mid W) = \frac{\lambda^r}{\det(\varphi \mid V)}$$